Optimization II

So far you have learned how to find the relative and absolute extrema of a function. This is an important concept because of how it can be applied to real life situations. In many situations you will be looking to either minimize or maximize a quantity; whether it is time, cost, units produced, volume, etc. Probably the most used application of extrema in the business world is to minimize costs or to maximize revenue and profits.

In most of the cases, these application problems will be in the form of a word problem. Therefore, here are some helpful steps in solving these problems.

1. Read through the <u>entire</u> problem carefully before attempting to solve it.
2. Once you have read the problem, identify the information that is given and what you are being asked to find.
3. Sketching a diagram of the problem can sometimes help to visualize what needs to be done and the equation to be used.
4. Choose a variable to represent your unknown quantity.
5. Use the relationship between the known and unknown quantities to develop the function to be minimized or maximized. Be sure to locate any domain restrictions, it they exist.
6. Find the critical numbers of the function.
7. Find the extrema for the function.

Now lets look at a few examples of the applications of extrema.

Example 1: Find two nonnegative numbers x and y for which $x + y = 150$, such that x^2y is maximized.

Solution:

Step 1: Identify given information and what needs to be determined.

We are given that x and y are nonnegative numbers (therefore $x \geq 0$ and $y \geq 0$) and that their sum is 150 ($x + y = 150$).

We are asked to find the values of x and y that maximize the equation x^2y. To do this, we will assign a variable to this equation, such as $M = x^2y$.

Example 1 (Continued):

Step 2: Express the function to be maximized with only a single variable and determine the domain of the function

To maximize the equation $M = x^2y$, we want to rewrite it so that it contains only a single variable (either x or y). To do this, we will use the given equation of $x + y = 150$ and solve it for either x or y.

$$x + y = 150$$
$$y = 150 - x$$

Now we can substitute this into the equation $M = x^2y$

$$\begin{aligned} M &= x^2y \\ &= x^2(150 - x) \\ &= 150x^2 - x^3 \end{aligned}$$

Determine the domain of the function

Since x & y are nonnegative numbers $M \geq 0$

$$150x^2 - x^3 \geq 0$$
$$x^2(150 - x) \geq 0$$
$$x^2 \geq 0 \quad \text{or} \quad 150 - x \geq 0$$
$$x \geq 0 \qquad\qquad -x \geq -150$$
$$x \leq 150$$

Therefore, the domain is $0 \leq x \leq 150$

Step 3: Find the critical numbers of the function

First find the derivative

$$M = 150x^2 - x^3$$
$$M' = 300x - 3x^2$$

$$M' = 300x - 3x^2$$
$$0 = 300x - 3x^2$$
$$0 = 3x(100 - x)$$

$$3x = 0 \qquad\qquad 100 - x = 0$$
$$x = 0 \qquad\qquad 100 = x$$

Example 1 (Continued):

Both numbers (0 and 100) are within our restricted domain ($0 \le x \le 150$) so both are valid critical numbers.

Step 4: Determine the maximum value of the function

Evaluate the function $M = 150x^2 - x^3$ at the critical numbers (0 & 100) and endpoints (0 & 150) of the domain.

If $x = 0$

$$\begin{aligned} M &= 150x^2 - x^3 \\ &= 150(0)^2 - (0)^3 \\ &= 0 \end{aligned}$$

If $x = 100$

$$\begin{aligned} M &= 150x^2 - x^3 \\ &= 150(100)^2 - (100)^3 \\ &= 1{,}500{,}000 - 1{,}000{,}000 \\ &= 500{,}000 \end{aligned}$$

If $x = 150$

$$\begin{aligned} M &= 150x^2 - x^3 \\ &= 150(150)^2 - (150)^3 \\ &= 3{,}375{,}000 - 3{,}375{,}000 \\ &= 0 \end{aligned}$$

The maximum value is 500,000, which occurs when $x = 100$. So now we can find y. (Remember the problem asked you to find the values for x **and** y)

$$\begin{aligned} y &= 150 - x \\ &= 150 - 100 \\ &= 50 \end{aligned}$$

The solution that maximizes the equation $M = x^2y$ is $x = 100$ and $y = 50$.

Example 2: A new communicable disease is spreading in a Texas city. It is estimated that *t* days after the disease is first observed the percent of the population infected can be approximated by the formula

$$p(t) = \frac{20t^3 - t^4}{1000} \qquad \text{for } 0 \le t \le 20.$$

a) After how many days is the percent of the population infected a maximum?
b) What is the maximum percent of the population infected?

Solution:

Step 1: Identify given information and what needs to be determined.

Given information:

t = number of days after disease is first observed

$p(t) = \frac{20t^3 - t^4}{1000}$ formula for approximating population infected

$0 \le t \le 20$ is the domain restriction for the formula

What needs to be determined:

The value of t that will maximize the function $p(t)$ and the maximum percent of the population infected.

Step 2: Express the function to be maximized with only a single variable and determine the domain of the function.

Since we were already given the formula in terms of just a single variable (t) and the domain ($0 \le t \le 20$) we can proceed to the next step.

Example 2 (Continued):

Step 3: Find the critical numbers of the function

First we must find the derivative of $p(t)$. If we split the function into two separate fractions before finding the derivative we can reduce the function to a power function and avoid having to use the quotient rule.

$$
\begin{aligned}
p(t) &= \frac{20t^3 - t^4}{1000} \\
&= \frac{20}{1000}t^3 - \frac{1}{1000}t^4 \\
p'(t) &= \frac{20}{1000}\left(3t^2\right) - \frac{1}{1000}\left(4t^3\right) \\
&= \frac{60t^2}{1000} - \frac{4t^3}{1000} \\
&= \frac{60t^2 - 4t^3}{1000}
\end{aligned}
$$

Find where the derivative is zero or undefined

Since the denominator is a constant number (1000) it will never be equal to zero and therefore the derivative will never be undefined. So our only critical numbers will come from when the numerator is equal to zero.

$$
\begin{aligned}
60t^2 - 4t^3 &= 0 \\
4t^2\left(15 - t\right) &= 0
\end{aligned}
$$

$$
\begin{aligned}
4t^2 &= 0 & \qquad 15 - t &= 0 \\
t^2 &= 0 & -t &= -15 \\
t &= 0 & t &= 15
\end{aligned}
$$

Both of these numbers are within our restricted domain $0 \le t \le 20$ so they are both valid critical numbers.

Example 2 (Continued):

Step 4: Determine the maximum value of the function

Evaluate the function at the critical numbers (0 & 15) and the endpoints (0 & 20) of the domain.

If x = 0

$$p(t) = \frac{20t^3 - t^4}{1000}$$
$$p(0) = \frac{20(0)^3 - (0)^4}{1000}$$
$$= \frac{0}{1000}$$
$$= 0$$

If x = 15

$$p(t) = \frac{20t^3 - t^4}{1000}$$
$$p(15) = \frac{20(15)^3 - (15)^4}{1000}$$
$$= \frac{67500 - 50625}{1000}$$
$$= \frac{16875}{1000}$$
$$= 16.875$$

If x = 20

$$p(t) = \frac{20t^3 - t^4}{1000}$$
$$p(20) = \frac{20(20)^3 - (20)^4}{1000}$$
$$= \frac{160000 - 160000}{1000}$$
$$= \frac{0}{1000}$$
$$= 0$$

Example 2 (Continued):

The maximum value is 16.875 when x = 15. Therefore, the infection of the population will reach its maximum percentage of **16.875%** after **15** days

Example 3: The total profit (in tens of dollars) from the sale of x hundred boxes of candy is given by $P(x) = -x^3 + 11x^2 - 18x$.

a) Find the number of boxes of candy that should be sold in order to produce maximum profit.
b) Find the maximum profit.

Solution:

Step 1: Identify given information and what needs to be determined.

Given information:

x = boxes of candy sold (in hundreds)
$P(x) = -x^3 + 11x^2 - 18x$ is the profit equation
$x \geq 0$ (x is the number of boxes sold and must be a nonnegative number since it is impossible to sell a negative amount of boxes)

What needs to be determined:

The number of boxes (x) that must be sold to maximize the profit function and the maximum profit P(x).

Step 2: Express the function to be maximized with only a single variable and determine the domain of the function

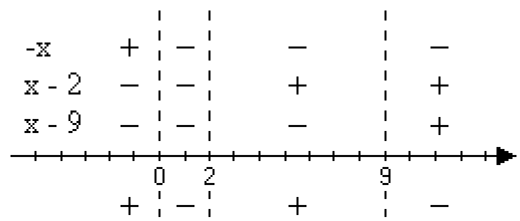
We are given the function for the profit is already but we are not given the domain. To find the domain, we will factor the given equation to find the zeros.

$$P(x) = -x^3 + 11x^2 - 18x$$
$$P(x) = -x(x^2 - 11x + 18)$$
$$P(x) = -x(x - 2)(x - 9)$$

$-x = 0$ or $x - 2 = 0$ or $x - 9 = 0$
$x = 0$ $x = 2$ $x = 9$

Example 3 (Continued):

Since we want to maximize the profit we want to determine where it is positive.



The only positive intervals for the profit function are x ≤ 0 and 2 ≤ x ≤ 9. However, since we cannot sell a negative number of boxes x cannot be less than zero so our domain would be 2 ≤ x ≤ 9.

Step 3: Find the critical numbers of the function

First, find the derivative of P(x)

$$P(x) = -x^3 + 10x^2 - 12x$$
$$P'(x) = -3x^2 + 20x - 12$$

Now set the derivative equal to zero and solve for x.

$$P'(x) = -3x^2 + 20x - 12$$
$$0 = -3x^2 + 20x - 12$$
$$0 = (-3x + 2)(x - 6)$$

$$-3x + 2 = 0 \qquad x - 6 = 0$$
$$-3x = -2 \qquad x = 6$$
$$x = \frac{2}{3}$$

The only number in our domain is x = 6. We would ignore 2/3 as a possible critical number for the absolute maximum.

Example 3 (Continued):

Step 4: Determine the maximum value of the function

If x = 2

$$\begin{aligned} P(x) &= -x^3 + 11x^2 - 18x \\ P(2) &= -(2)^3 + 11(2)^2 - 18(2) \\ &= -8 + 44 - 36 \\ &= 0 \end{aligned}$$

If x = 6

$$\begin{aligned} P(x) &= -x^3 + 10x^2 - 12x \\ P(6) &= -(6)^3 + 10(6)^2 - 12(6) \\ &= -216 + 360 - 72 \\ &= 72 \end{aligned}$$

If x = 9

$$\begin{aligned} P(x) &= -x^3 + 11x^2 - 18x \\ P(9) &= -(9)^3 + 11(9)^2 - 18(9) \\ &= -729 + 891 - 162 \\ &= 0 \end{aligned}$$

The largest value we get for the profit is 72 when x is 6. Since the profit is in tens of dollars and x is the boxes sold in hundreds, the maximum profit will be **$720** when **600** boxes are sold.

Example 4: A manufacturing company needs to design an open-topped box with a square base. The box must have a volume of 32 in^3. Find the dimensions of the box that can be built with the minimum amount of materials.

Solution:

Step 1: Identify given information and what needs to be determined.

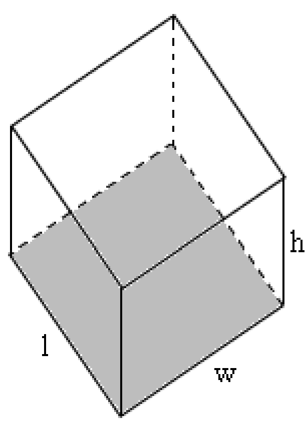
Given information:

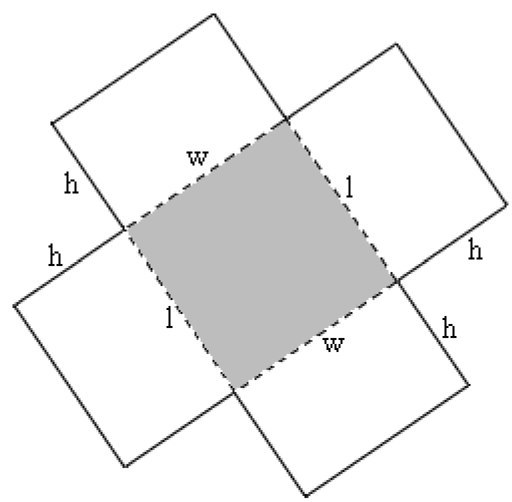
The box does not have a top and the base is a square
The volume of the box must be 32 in^3

What needs to be determined:

The dimensions (l, w, h) that will minimize the amount of materials (surface area) need

Visual representation:





Step 2: Express the function to be minimized with only a single variable and determine the domain of the function

Our goal is to minimize the material used so we want to minimize the surface area of the box. By looking at the collapsed view of the box we can see that the surface area will be made up of the square base and the four sides. Since the base is a square the length and width must be the same. We will let x = the length/width and y = the height of the box.

Surface area of base: $l * w = x * x$
$= x^2$

Example 4 (Continued):

Surface area of side: $l * h = x * y$
$= xy$

Total surface area = surface area of base + 4(surface area of side)
$A = x^2 + 4xy$

Now we must rewrite this equation so that it has only one variable. To do this, we will use the equation for the volume of a box and the given volume.

$$V = l * w * h$$
$$32 = x * x * y$$
$$32 = x^2 y$$
$$\frac{32}{x^2} = y$$

Substitute the expression for y into our area equation.

$$A = x^2 + 4x\left(\frac{32}{x^2}\right)$$
$$A = x^2 + \frac{128}{x}$$

The only restriction on x is that it must be greater than zero ($x > 0$).

Step 3: Find the critical numbers of the function

First, find the derivative

$$A = x^2 + \frac{128}{x}$$
$$A = x^2 + 128x^{-1}$$
$$A' = 2x - 128x^{-2}$$
$$A' = 2x - \frac{128}{x^2}$$

Example 4 (Continued):

Find the critical numbers

$$\begin{aligned} A' &= 2x - \frac{128}{x^2} \\ &= \frac{2x^3}{x^2} - \frac{128}{x^2} \\ &= \frac{2x^3 - 128}{x^2} \end{aligned}$$

$$\begin{aligned} 2x^3 - 128 &= 0 \\ 2x^3 &= 128 \\ x^3 &= 64 \\ x &= 4 \end{aligned} \qquad \begin{aligned} x^2 &= 0 \\ x &= 0 \end{aligned}$$

Step 4: Determine the minimum value of the function

If x = 4

$$\begin{aligned} A &= x^2 + \frac{128}{x} \\ &= (4)^2 + \frac{128}{4} \\ &= 16 + 32 \\ &= 48 \end{aligned}$$

$$\begin{aligned} y &= \frac{32}{x^2} \\ &= \frac{32}{4^2} \\ &= \frac{32}{16} \\ &= 2 \end{aligned}$$

In order to minimize the surface area of the box the **length** and **width** should be **4 in** and the **height** should be **2 in**.