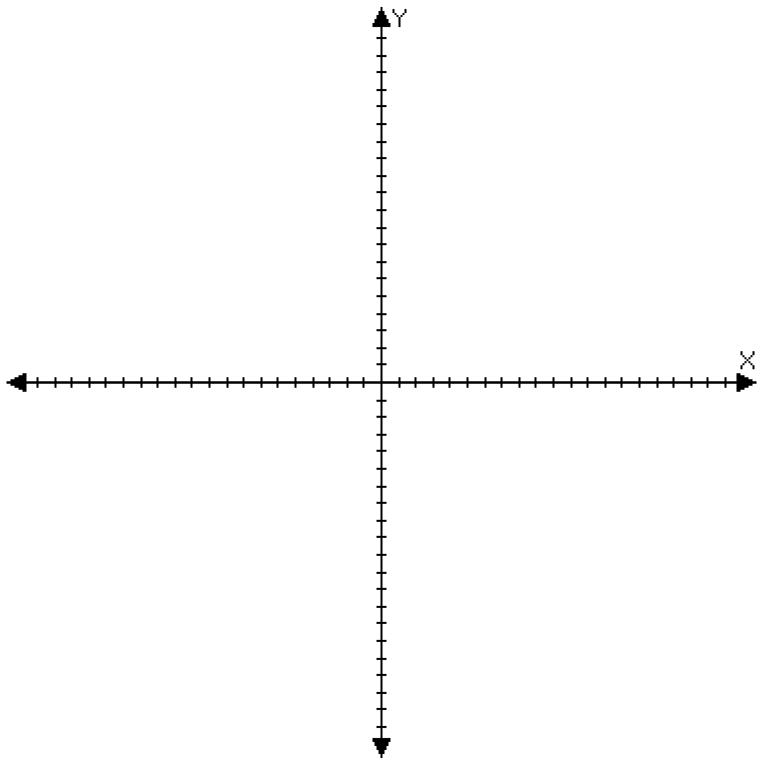
Review Exercise Set 14

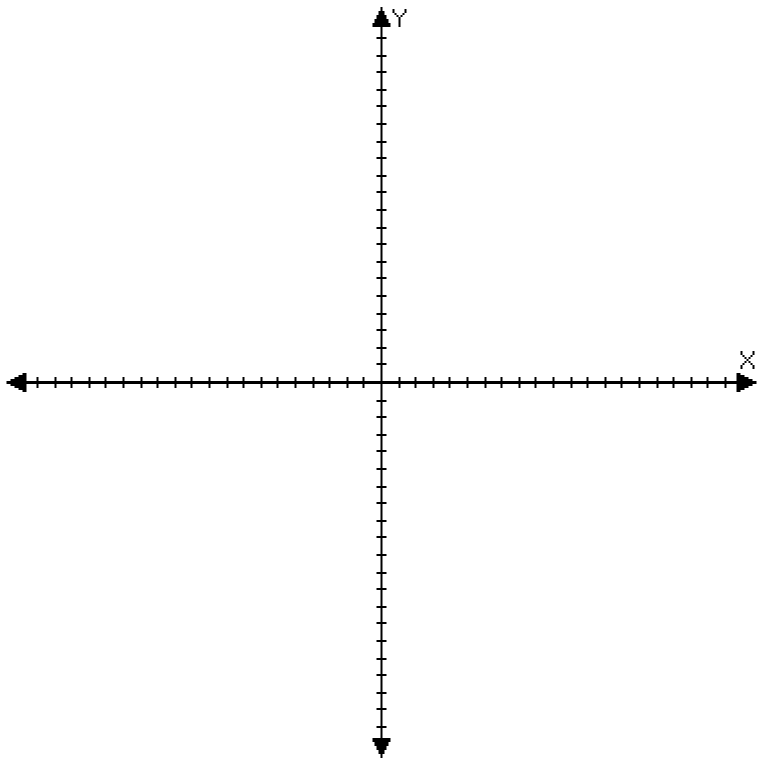
Exercise 1: Graph the given function by finding the domain, critical points, intervals where the function is increasing or decreasing, points of inflection, intervals where the function is concave up or down, any intercepts or asymptotes.

$g(x) = -3x^3 + 6x^2 - 4x - 1$



Exercise 2: Graph the given function by finding the domain, critical points, intervals where the function is increasing or decreasing, points of inflection, intervals where the function is concave up or down, any intercepts or asymptotes.

$$y = (x + 1)^2 / (1 + x^2)$$



Exercise 3: Sketch the graph of a function that has all of the following properties.

Continuous and differentiable for all values of x except -3

$\lim_{x \to -3^-} f(x) = \infty$ and $\lim_{x \to -3^+} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = 1$ and $\lim_{x \to \infty} f(x) = 1$

x-intercept at x = -2 and y-intercept at y = 4

f '(x) > 0 on the intervals (- ∞, -3) and (-3, 2)

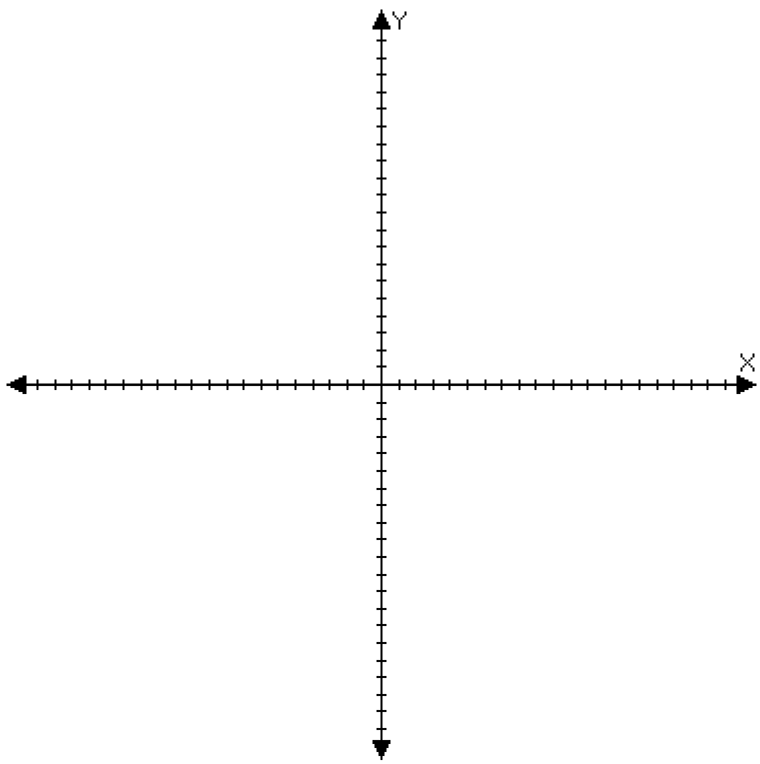
f '(x) < 0 on the interval (2, ∞)

f '(2) = 0

f "(x) > 0 on the intervals (- ∞, -3) and (4, ∞)

f "(x) > 0 on the interval (-3, 4)

Point of inflection at (4, 3)



Review Exercise Set 14 Answer Key

Exercise 1: Graph the given function by finding the domain, critical points, intervals where the function is increasing or decreasing, points of inflection, intervals where the function is concave up or down, any intercepts or asymptotes.

$g(x) = -3x^3 + 6x^2 - 4x - 1$

Domain: All real numbers

First derivative

$g'(x) = -9x^2 + 12x - 4$

Critical numbers of first derivative

$0 = -9x^2 + 12x - 4$
$0 = -(9x^2 - 12x + 4)$
$0 = -(3x - 2)(3x - 2)$
$0 = -(3x - 2)^2$
$3x - 2 = 0$
$x = \frac{2}{3}$

Increasing/decreasing intervals

	$(-\infty, 2/3)$	$(2/3, \infty)$
-1	-	-
$(3x - 2)^2$	+	+
f'(x)	-	-

Increasing: none
Decreasing: $(-\infty, 2/3) \cup (2/3, \infty)$

Second derivative

$f''(x) = -18x + 12$

Critical numbers of second derivative

$0 = -18x + 12$
$0 = -6(3x - 2)$
$3x - 2 = 0$
$x = \frac{2}{3}$

Exercise 1 (Continued):

Concave up/down intervals

	$(-\infty,\ 2/3)$	$(2/3, \infty)$
-6	-	-
(3x - 2)	-	+
f''(x)	+	-

Concave up: $(-\infty,\ 2/3)$

Concave down: $(2/3, \infty)$

Points of inflection

$g(x) = -3x^3 + 6x^2 - 4x - 1$

$g(\frac{2}{3}) = -3(\frac{2}{3})^3 + 6(\frac{2}{3})^2 - 4(\frac{2}{3}) - 1$

$= -\frac{8}{9} + \frac{8}{3} - \frac{8}{3} - 1$

$= -\frac{17}{9}$

Points of inflection: $\left(\frac{2}{3}, -\frac{17}{9}\right)$

Intercepts

y-intercept:

$g(x) = -3x^3 + 6x^2 - 4x - 1$

$g(0) = -3(0)^3 + 6(0)^2 - 4(0) - 1$

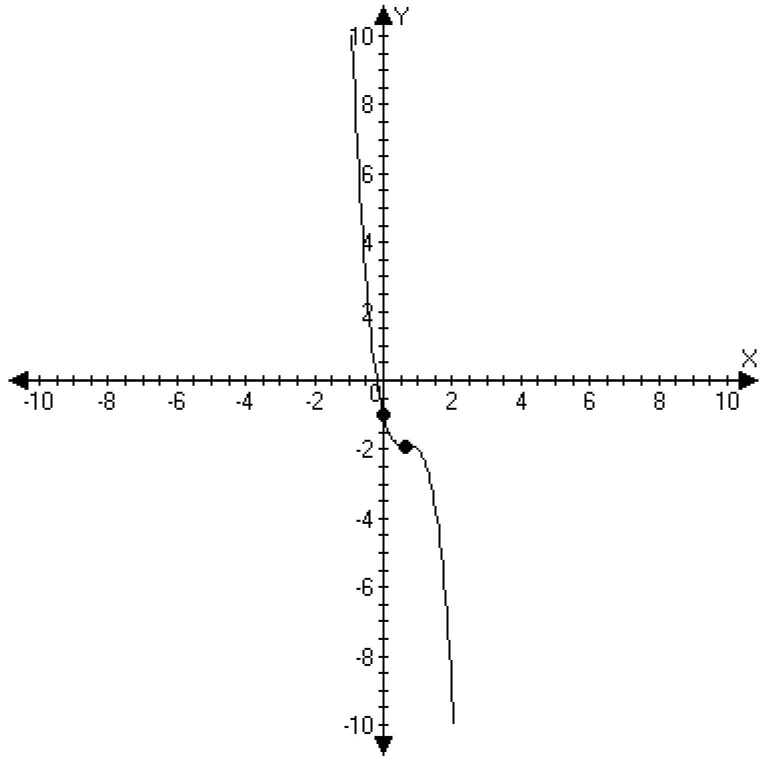
$g(0) = -1$

(0, -1)

Asymptotes: None

Exercise 1 (Continued):

Graph



Exercise 2: Graph the given function by finding the domain, critical points, intervals where the function is increasing or decreasing, points of inflection, intervals where the function is concave up or down, any intercepts or asymptotes.

$y = (x + 1)^2 / (1 + x^2)$

Domain:

$1 + x^2 = 0$
$x^2 = -1$

x^2 cannot be negative, so the domain is all real numbers

First derivative

$D_x (x + 1)^2 = 2(x + 1)$

$D_x(1 + x^2) = 2x$

Exercise 2 (Continued):

$$y' = \frac{\left(1+x^2\right)D_x\left(x+1\right)^2 - \left(x+1\right)^2 D_x\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$= \frac{\left(1+x^2\right)\left(2\right)\left(x+1\right)-\left(x+1\right)^2\left(2x\right)}{\left(1+x^2\right)^2}$$

$$= \frac{2\left(x+1\right)\left[\left(1+x^2\right)-\left(x+1\right)\left(x\right)\right]}{\left(1+x^2\right)^2}$$

$$= \frac{2\left(x+1\right)\left(1-x\right)}{\left(1+x^2\right)^2}$$

$$= \frac{-2\left(x+1\right)\left(x-1\right)}{\left(1+x^2\right)^2}$$

Critical numbers of first derivative

numerator = 0

0 = -2(x + 1)(x - 1)
x + 1 = 0 or x - 1 = 0
x = -1 or x = 1

denominator = 0

$0 = (1 + x^2)^2$
$0 = 1 + x^2$
$-1 = x^2$
x^2 cannot be negative

Increasing/decreasing intervals

	$\left(-\infty,-1\right)$	$\left(-1,1\right)$	$\left(1,\infty\right)$
-2	-	-	-
(x + 1)	-	+	+
(x - 1)	-	-	+
$(1 + x^2)^2$	+	+	+
y'	-	+	-

Increasing: $\left(-1,1\right)$

Decreasing: $\left(-\infty,-1\right)\cup\left(1,\infty\right)$

Exercise 2 (Continued):

Second derivative

$$y' = \frac{-2(x+1)(x-1)}{(1+x^2)^2}$$
$$= \frac{-2x^2+2}{(1+x^2)^2}$$

$D_x(-2x^2 + 2) = -4x$

$D_x(1 + x^2)^2 = 2(1 + x^2)(2x) = 4x(1 + x^2)$

$$y'' = \frac{(1+x^2)^2 D_x(-2x^2+2) - (-2x^2+2) D_x(1+x^2)^2}{\left[(1+x^2)^2\right]^2}$$
$$= \frac{(1+x^2)^2(-4x) - (-2x^2+2)(4x)(1+x^2)}{(1+x^2)^4}$$
$$= \frac{(4x)(1+x^2)\left[(1+x^2)(-1) - (-2x^2+2)\right]}{(1+x^2)^4}$$
$$= \frac{(4x)(x^2-3)}{(1+x^2)^3}$$

Critical numbers of second derivative

The denominator cannot be zero so set the numerator equal to zero

$0 = 4x(x^2 - 3)$
$4x = 0$ or $x^2 - 3 = 0$
$x = 0$ or $x^2 = 3$
$x = 0$ or $x = \pm\sqrt{3}$

Concave up/down intervals

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
4x	-	-	+	+
$(x^2 - 3)$	+	-	-	+
$(1 + x^2)^3$	+	+	+	+
f''(x)	-	+	-	+

Exercise 2 (Continued):

Concave up: $\left(-\sqrt{3},0\right)\cup\left(\sqrt{3},\infty\right)$

Concave down: $\left(-\infty,-\sqrt{3}\right)\cup\left(0,\sqrt{3}\right)$

Points of inflection

$$\begin{aligned} y &= \frac{(x+1)^2}{1+x^2} \\ &= \frac{\left(-\sqrt{3}+1\right)^2}{1+\left(-\sqrt{3}\right)^2} \\ &= \frac{\left(-\sqrt{3}+1\right)^2}{4} \\ &\approx 0.134 \end{aligned}$$

$$\begin{aligned} y &= \frac{(x+1)^2}{1+x^2} \\ &= \frac{(0+1)^2}{1+(0)^2} \\ &= 1 \end{aligned}$$

$$\begin{aligned} y &= \frac{(x+1)^2}{1+x^2} \\ &= \frac{\left(\sqrt{3}+1\right)^2}{1+\left(\sqrt{3}\right)^2} \\ &= \frac{\left(\sqrt{3}+1\right)^2}{4} \\ &\approx 1.866 \end{aligned}$$

Points of inflection: (-$\sqrt{3}$, 0.134), (0, 1), and ($\sqrt{3}$, 1.866)

Intercepts

y-intercept: (0, 1)

x-intercept:

$$y = \frac{(x+1)^2}{1+x^2}$$

$$0 = \frac{(x+1)^2}{1+x^2}$$

$0 = (x + 1)^2$

0 = x + 1

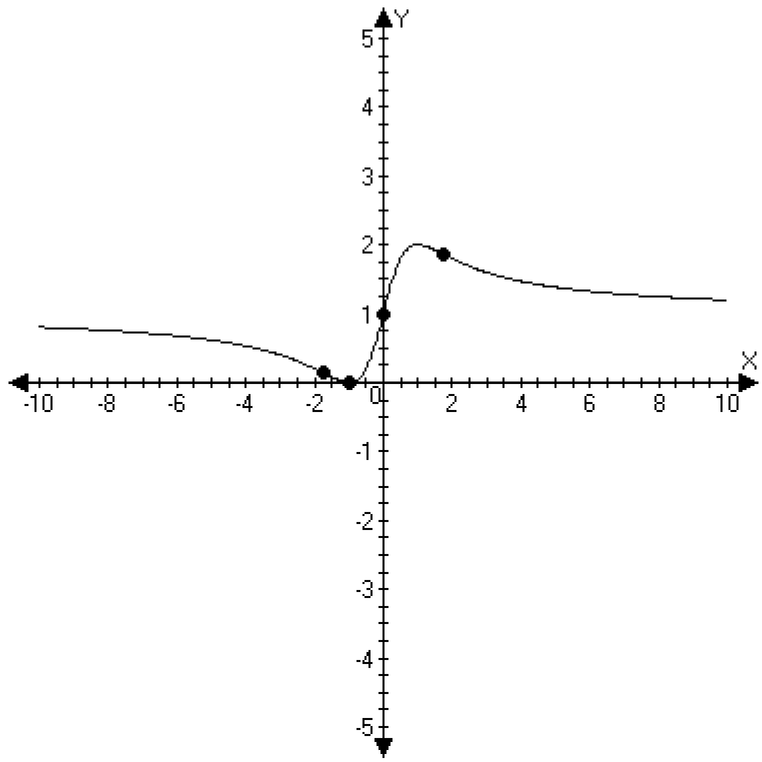
-1 = x

(-1, 0)

Asymptotes: None

Exercise 2 (Continued):

Graph



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Point of inflection at (4, 3)

Exercise 3 (Continued):