Implicit Differentiation and Related Rates

Up until now you have been finding the derivatives of functions that have already been solved for their dependent variable. However, there are some functions that cannot be easily solved for the dependent variable so we need to have a way of still finding the derivative. This process is called implicit differentiation.

Finding the derivative of a function by implicit differentiation uses the same derivative formulas that were covered earlier. The important part to remember is that when you take the derivative of the dependent variable you must include the derivative notation dy/dx or y' in the derivative. The notation that is used depends on which is easier for you.

Let's take the function 2xy + 3x = 11 as an example. This function can easily be solved for the dependent variable y, but lets look at how implicit differentiation works.

The first term 2xy is the product of 2x and y so we would apply the product rule. First we would take the derivative of each term and then substitute into the product rule.

$$\frac{d}{dx}(2x) = 2 \qquad \frac{d}{dx}(y) = 1 \cdot \frac{dy}{dx} = \frac{dy}{dx} \text{ or } y'$$

$$\begin{aligned}\frac{d}{dx}(2xy) &= (2x)(y') + (y)(2) \\ &= 2xy' + 2y\end{aligned}$$

Taking the derivatives of "3x" and "11" would be done in the same manner as before. So the implicit derivative would be:

$$\begin{aligned}2xy + 3x &= 11 \\ \frac{d}{dx}(2xy) + \frac{d}{dx}(3x) &= \frac{d}{dx}(11) \\ 2xy' + 2y + 3 &= 0\end{aligned}$$

Now we can solve for y′

$$\begin{aligned}2xy' + 2y + 3 &= 0 \\ 2xy' &= -2y - 3 \\ y' &= \frac{-2y - 3}{2x}\end{aligned}$$

Therefore, the steps involved in using implicit differentiation to find dy/dx are:

1. Differentiate each term on both sides of the equals sign with respect to the independent variable "x". When taking the derivative of the dependent variable "y" don't forget to include dy/dx or y′ in the derivative.
2. Solve for dy/dx or y′ by grouping only the terms with dy/dx or y′ on one side of the equals sign
3. Factor out dy/dx or y′
4. Divide to isolate dy/dx or y′

Let's look at a few examples.

Example 1: Find the derivative of $x^2y^3 + 4xy = 2x$ using implicit differentiation.

Solution:

Step 1: Differentiate each term on both sides of the equals sign

$$x^2y^3 + 4xy = 2x$$

$$\frac{d}{dx}\left(x^2y^3\right) + \frac{d}{dx}\left(4xy\right) = \frac{d}{dx}\left(2x\right)$$

Apply product rule to both terms on the left

$$\left[\left(x^2\right)\frac{d}{dx}\left(y^3\right) + \left(y^3\right)\frac{d}{dx}\left(x^2\right)\right] + \left[\left(4x\right)\frac{d}{dx}\left(y\right) + \left(y\right)\frac{d}{dx}\left(4x\right)\right] = 2$$

Apply the chain rule

$$\left[\left(x^2\right)\cdot(3)\left(y^2\right)\frac{dy}{dx} + \left(y^3\right)\cdot 2x\right] + \left[\left(4x\right)\cdot(1)\frac{dy}{dx} + \left(y\right)\cdot(4)\right] = 2$$

Step 2: Group all terms with dy/dx on one side

$$3x^2y^2\frac{dy}{dx} + 2xy^3 + 4x\frac{dy}{dx} + 4y = 2$$

$$3x^2y^2\frac{dy}{dx} + 4x\frac{dy}{dx} = 2 - 2xy^3 - 4y$$

Step 3: Factor dy/dx

$$\frac{dy}{dx}\left(3x^2y^2 + 4x\right) = 2 - 2xy^3 - 4y$$

Example 1 (Continued):

Step 4: Divide to isolate dy/dx

$$\frac{dy}{dx}=\frac{2-2xy^3-4y}{3x^2y^2+4x}$$

Example 2: Find the derivative of $\frac{4y+5x}{1-3y}=\frac{1}{6x}$ using implicit differentiation.

Solution:

Step 1: Differentiate each term on both sides of the equals sign

$$\frac{d}{dx}\left(\frac{4y+5x}{1-3y}\right)=\frac{d}{dx}\left(\frac{1}{6x}\right)$$

We can simplify the right side by using the properties of exponents

$$\frac{d}{dx}\left(\frac{4y+5x}{1-3y}\right)=\frac{d}{dx}(6x)^{-1}$$

Apply quotient rule to the left side and the generalized power rule to the right side

$$\frac{d}{dx}\left(\frac{4y+5x}{1-3y}\right)=\frac{d}{dx}(6x)^{-1}$$

$$\frac{(1-3y)\frac{d}{dx}(4y+5x)-(4y+5x)\frac{d}{dx}(1-3y)}{(1-3y)^2}=(-1)(6x)^{-2}\frac{d}{dx}(6x)$$

$$\frac{(1-3y)\left(4\frac{dy}{dx}+5\right)-(4y+5x)\left(-3\frac{dy}{dx}\right)}{(1-3y)^2}=(-1)(6x)^{-2}(6)$$

In this example we will replace the dy/dx notation with y′

$$\frac{(1-3y)(4y'+5)-(4y+5x)(-3y')}{(1-3y)^2}=(-1)(6x)^{-2}(6)$$

Example 2 (Continued):

Step 2: Group all terms with y′ on one side

$$\frac{(1-3y)(4y'+5)-(4y+5x)(-3y')}{(1-3y)^2}=(-1)(6x)^{-2}(6)$$

$$\frac{(4y'+5-12yy'-15y)-(-12yy'-15xy')}{(1-3y)^2}=-\frac{6}{(6x)^2}$$

$$\frac{4y'+5-12yy'-15y+12yy'+15xy'}{(1-3y)^2}=-\frac{6}{36x^2}$$

$$\frac{4y'+5-15y+15xy'}{(1-3y)^2}=-\frac{6}{36x^2}$$

$$4y'+5-15y+15xy'=-\frac{6(1-3y)^2}{36x^2}$$

$$4y'+15xy'=-\frac{6(1-3y)^2}{36x^2}+15y-5$$

Step 3: Factor y′

$$y'(4+15x)=-\frac{6(1-3y)^2}{36x^2}+\frac{36x^2(15y-5)}{36x^2}$$

Step 4: Divide to isolate y′

$$y'=\frac{1}{4+15x}\left(\frac{-6(1-3y)^2+36x^2(15y-5)}{36x^2}\right)$$

$$y'=\frac{-6(1-3y)^2+36x^2(15y-5)}{36x^2(4+15x)}$$

We can also find the equation of a tangent line at a given point using implicit differentiation. The steps involved in do this are:

1. Find the derivative using implicit differentiation
2. If both the x and y coordinates are not known find the missing coordinate
3. Substitute the x and y coordinates into the derivative to find the slope of the tangent line
4. Find the equation of the tangent line using the point-slope formula

Example 3: Find the equation of the tangent line when x = 4 for the curve $y+\frac{\sqrt{x}}{y}=3$.

Solution:

Step 1: Find the derivative y′ using implicit differentiation

$$y+\frac{\sqrt{x}}{y}=3$$

$$\frac{d}{dx}(y)-\frac{d}{dx}\left(\frac{x^{1/2}}{y}\right)=\frac{d}{dx}(3)$$

$$y'-\frac{y\left(\frac{1}{2}x^{-1/2}\right)-x^{1/2}y'}{y^2}=0$$

$$y'-\frac{y\left(\frac{1}{2x^{1/2}}\right)-x^{1/2}y'}{y^2}=0$$

$$y^2\left(y'-\frac{y\left(\frac{1}{2x^{1/2}}\right)-x^{1/2}y'}{y^2}=0\right)$$

$$y^2y'-\left(\frac{y}{2x^{1/2}}-x^{1/2}y'\right)=0$$

$$y^2y'-\frac{y}{2x^{1/2}}+x^{1/2}y'=0$$

$$y^2y'+x^{1/2}y'=\frac{y}{2x^{1/2}}$$

$$y'\left(y^2+x^{1/2}\right)=\frac{y}{2x^{1/2}}$$

$$y'=\frac{y}{2x^{1/2}\left(y^2+x^{1/2}\right)}$$

$$y'=\frac{y}{2x^{1/2}y^2+2x}$$

Example 3 (Continued):

Step 2: Find any missing coordinates.

In this problem we know the value for x but not y. So we must find the corresponding y value when x = 2.

$$y+\frac{\sqrt{x}}{y}=3$$

$$y+\frac{\sqrt{4}}{y}=3$$

$$y+\frac{2}{y}=3$$

$$y\left(y+\frac{2}{y}=3\right)$$

$$y^2+2=3y$$

$$y^2-3y+2=0$$

$$(y-1)(y-2)=0$$

$$y-1=0 \quad or \quad y-2=0$$

$$y=1 \qquad\qquad y=2$$

Step 3: Find slope of tangent line

x = 4, y = 1

$$\begin{aligned} y' &= \frac{y}{2y^2\sqrt{x}+2x} \\ &= \frac{1}{2(1)^2\sqrt{4}+2(4)} \\ &= \frac{1}{2(1)(2)+8} \\ &= \frac{1}{4+8} \\ &= \frac{1}{12} \end{aligned}$$

x = 4, y = 2

$$\begin{aligned} y' &= \frac{y}{2y^2\sqrt{x}+2x} \\ &= \frac{2}{2(2)^2\sqrt{4}+2(4)} \\ &= \frac{2}{2(2)(2)+8} \\ &= \frac{2}{8+8} \\ &= \frac{2}{16} \\ &= \frac{1}{8} \end{aligned}$$

Example 3 (Continued):

Step 4: Find the equation of the tangent lines

x = 4, y = 1, y′ = 1/12

$$y - y_1 = m(x - x_1)$$
$$y - 1 = \frac{1}{12}(x - 4)$$
$$y - 1 = \frac{1}{12}x - \frac{1}{3}$$
$$y = \frac{1}{12}x - \frac{1}{3} + 1$$
$$y = \frac{1}{12}x + \frac{2}{3}$$

x = 4, y = 2, y′ = 1/8

$$y - y_1 = m(x - x_1)$$
$$y - 2 = \frac{1}{8}(x - 4)$$
$$y - 2 = \frac{1}{8}x - \frac{1}{2}$$
$$y = \frac{1}{8}x - \frac{1}{2} + 2$$
$$y = \frac{1}{8}x + \frac{3}{2}$$

Another application for implicit differentiation is the topic of related rates. Related rates are used to determine the rate at which a variable is changing with respect to time. We use the concept of implicit differentiation because time is not usually a variable in the equation.

For example, if we were asked to determine the rate at which the area of a square is changing then implicit differentiation must be used because the equation for the area of a square only contains the variables for the length, width, and area. Time is not a variable in the equation so the only way to determine the rate at which the area is changing (dA/dt) is to take the derivative implicitly.

The steps involved in solving a related rates problem can be summarized as:

1. Identify all given information and what we must find.
2. Draw a sketch if it is possible
3. Determine the equation that relates the variables
4. Find the derivative using implicit differentiation
5. Solve the derivative for the unknown rate
6. Substitute in the given information and solve

Example 4: A 50-ft ladder is placed against a building. The top of the ladder is sliding down the building at the rate of 2 ft/min. Find the rate at which the base of the ladder is moving away from the building at the instant that the base is 30 ft from the building.

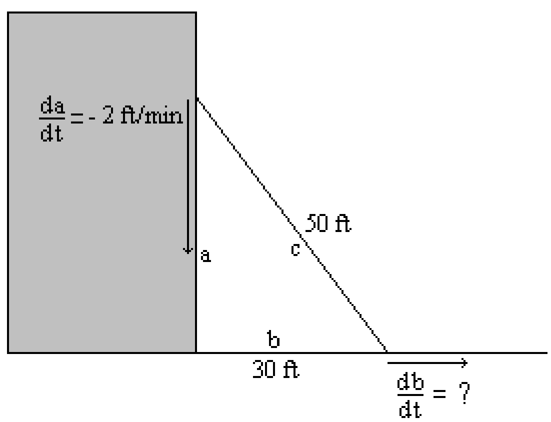
Solution:

Step 1: Identify all given information and what we must find

Length of ladder (c) = 50 ft
Initial height of ladder (a) = ?
Rate ladder is sliding down (da/dt) = - 2 ft/min*
Distance of base from building (b) = 30 ft
Rate base is moving away (db/dt) = ?

* The rate the ladder is sliding down is shown as a negative value to indicate the direction in which the top of the ladder is moving.

Step 2: Draw a sketch



Example 4 (Continued):

Step 3: Determine the equation that relates the variables

In this problem are variables are the sides the triangle formed by the ladder (c), the building (a), and the ground (b). An equation that would relate these three variables is the Pythagorean theorem, $a^2 + b^2 = c^2$.

Step 4: Find the derivative using implicit differentiation

$$a^2 + b^2 = c^2$$
$$\frac{d}{dt}\left(a^2 + b^2\right) = \frac{d}{dt}\left(c^2\right)$$
$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

Step 5: Solve the derivative for the unknown rate

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$
$$2b\frac{db}{dt} = 2c\frac{dc}{dt} - 2a\frac{da}{dt}$$
$$\frac{db}{dt} = \frac{2c\frac{dc}{dt} - 2a\frac{da}{dt}}{2b}$$

Step 6: Substitute in the given information and solve

Since "c" represents the length of the ladder and the length is not going to change the related rate dc/dt would be equal to zero. Also we do not know the value of "a" so we must find it first before finding db/dt.

$$a^2 + b^2 = c^2$$
$$a^2 + (30)^2 = (50)^2$$
$$a^2 + 900 = 2500$$
$$a^2 = 1600$$
$$a = \sqrt{1600}$$
$$a = 40$$

Example 4 (Continued):

Now we can solve for db/dt.

$$\frac{db}{dt} = \frac{2c\frac{dc}{dt} - 2a\frac{da}{dt}}{2b}$$
$$= \frac{2(50)(0) - 2(40)(-2)}{2(30)}$$
$$= \frac{160}{60}$$
$$= \frac{8}{3}$$

db/dt is approximately equal to 2.67 ft/min.

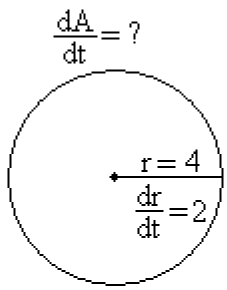
Example 5: A rock is thrown into a still pond. The circular ripples move outward from the point of impact of the rock so that the radius of the circle formed by a ripple increases at the rate of 2 ft per minute. Find the rate at which the area is changing at the instant the radius is 4 ft.

Solution:

Step 1: Identify all given information and what we must find

Rate the radius is changing (dr/dt) = 2 ft/min
Radius (r) = 4 ft
Rate the area of the circle is changing (dA/dt) = ?

Step 2: Draw a sketch



Step 3: Determine the equation that relates the variables

In this example we are dealing with the area of a circle, therefore the equation that we will use would be $A = \Pi r^2$

Example 5 (Continued):

Step 4: Find the derivative using implicit differentiation

$$A = \pi r^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Step 5: Solve the derivative for the unknown rate

The derivative is already solved for the unknown rate so we can go to the next step.

Step 6: Substitute in the given information and solve

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$= 2\pi(4)(2)$$

$$= 16\pi$$

dA/dt is approximately equal to 50.3 ft/min.

Example 6: The revenue function for a company is $60x - .5x^2$ and its cost function is $8x + 12$, where x is the daily production and sales. If the daily production is currently 20 units and the rate of change for production is 6 units per day, find the rate at which the company's profit is changing.

Solution:

Step 1: Identify all given information and what we must find

Revenue function $R(x) = 60x - .5x^2$
Cost function $C(x) = 8x + 12$
$x = 20$
$dx/dt = 6$
Profit function = ?
$dP/dt = ?$

Example 6 (Continued):

Step 2: Draw a sketch when it is possible

With this type of problem a sketch is not used.

Step 3: Determine the equation that relates the variables

Since we need to find the rate at which the profit is changing we need to find the Profit function. The profit function would be equal to the difference between the revenue and cost functions.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 60x - .5x^2 - (8x + 12) \\ &= 60x - .5x^2 - 8x - 12 \\ &= - .5x^2 + 52x - 12 \end{aligned}$$

Step 4: Find the derivative using implicit differentiation

$$\begin{aligned} P &= -.5x^2 + 52x - 12 \\ \frac{d}{dt}P &= \frac{d}{dt}\left(-.5x^2 + 52x - 12\right) \\ \frac{dP}{dt} &= -1x\frac{dx}{dt} + 52\frac{dx}{dt} - 0 \\ \frac{dP}{dt} &= -x\frac{dx}{dt} + 52\frac{dx}{dt} \end{aligned}$$

Step 5: Solve the derivative for the unknown rate

The equation is already solved for dP/dt.

Step 6: Substitute in the given information and solve

$$\begin{aligned} \frac{dP}{dt} &= -x\frac{dx}{dt} + 52\frac{dx}{dt} \\ &= -(20)(6) + 52(6) \\ &= -120 + 312 \\ &= 192 \end{aligned}$$