Integration by Substitution

In this section, you will learn how to use the substitution method to find the antiderivatives of more complex functions. As you may recall when you applied the Chain Rule, for example, to find the derivative of a composite function in the form of $f(g(x))$.

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) = f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

In the last section you learned how to find the antiderivative of functions in the forms such as x^n and b^x. However, what if you were asked to find the derivative of $(x^2 + 1)^4(2x)$. Right now your only option would be to expand the function into the polynomial form of $2x^9 + 8x^7 + 12x^5 + 8x^3 + 2x$ and then find the antiderivative as we did in the previous section. However, there is an easier way and this is where the substitution method comes in.

How the substitution method works is that you will simplify the composite function by letting a variable such as "u" equal the function $g(x)$. By doing this you can rewrite the composite function y = $f(g(x))$ as $y = f(u)$ since $u = g(x)$. Now lets find the differential of u (du).

$$u = g\left(x\right)$$

$$\frac{du}{dx} = g'\left(x\right)$$

$$du = g'\left(x\right)\cdot dx$$

Now you have all of the parts necessary to find the antiderivative of a composite function using integration by substitution. The antiderivative $\int f'\left(g\left(x\right)\right)\cdot g'\left(x\right)dx$ can now be simplified as follows:

$$F\left(x\right) = \int f'\underbrace{\left(g\left(x\right)\right)}_{\downarrow}\cdot\underbrace{g'\left(x\right)dx}_{\downarrow}$$

$$= \int f'(\quad u \quad)\cdot \quad du$$

As you may notice the composite function in the integrand has both an inside function "g" and an outside function f ′. When applying the substitution method you will let "u" equal the inside function. The inside function will be the inner part of a composite function, for example

$x^2 - 3$ is the inside function for $\left(x^2 - 3\right)^5$ and $x^3 - 1$ is the inside function for $\sqrt{x^3 - 1}$

Lets go back and look at the composite function $(x^2 + 1)^4(2x)$ and see how integration by substitution can be used to make finding the derivative easier.
In this example the inside function would be $x^2 + 1$ so you would set this equal to the variable u.

$u = x^2 + 1$

Next, you want to find the differential of *u*.

$$u = x^2 + 1$$
$$\frac{du}{dx} = 2x$$
$$du = 2x \cdot dx$$

Now lets perform the substitution and find the antiderivative.

$$\begin{aligned}\int \left(x^2+1\right)^4 2x \cdot dx &= \int (u)^4\, du \\ &= \frac{u^{4+1}}{4+1} + C \\ &= \frac{u^5}{5} + C \\ &= \frac{\left(x^2+1\right)^5}{5} + C\end{aligned}$$

When using the substitution method make sure to always substitute the inside function back into your answer for *u*. Do not leave your answer in terms of the substitution variable "*u*."

Another thing to watch out for when using the substitution method is to make sure the differential of *u* is actually present in the integrand. In the above example, we did not have a problem because *du* was equal to 2*x dx*, which was in the integrand of the original problem. However, there are cases in which this will not be so. Lets look at another example.

Example 1: Find the antiderivative of $\int \left(2x^3+5\right)^4 x^2 dx$ using integration by substitution.

Solution:

Step 1: Identify the inside function and set it equal to *u*.

In this example, the inside function is $2x^3 + 5$ since it is being raised to an exponent. Therefore, $u = 2x^3 + 5$.

Example 1 (Continued):

Step 2: Find the differential of *u*.

$$u = 2x^3 + 5$$
$$\frac{du}{dx} = 3 \cdot 2x^{3-1}$$
$$\frac{du}{dx} = 6x^2$$
$$du = 6x^2 \cdot dx$$

Step 3: Check if the differential of *u* is present in the integrand

$du = 6x^2 \cdot dx$ but the integrand only contains $x^2 \cdot dx$ so you must adjust the differential of u to match. This can be done by dividing both sides by 6.

$$\frac{du}{6} = \frac{6x^2 \cdot dx}{6}$$
$$\frac{1}{6} du = x^2 \cdot dx$$

Step 4: Perform the substitution and find the antiderivative.

$$\int \left(2x^3 + 5\right)^4 x^2 dx = \int u^4 \cdot \frac{1}{6} du$$
$$= \frac{1}{6} \int u^4 \cdot du$$
$$= \frac{1}{6}\left(\frac{u^5}{5}\right) + C$$
$$= \frac{1}{30} u^5 + C$$

Step 5: Substitute the inside function back into the antiderivative.

$$\int \left(2x^3 + 5\right)^4 x^2 dx = \frac{1}{30} u^5 + C$$
$$= \frac{1}{30}\left(2x^3 + 5\right)^5 + C$$

This substitution method can also be used for problems involving exponential functions (where the base is raised to a function) and rational functions (where the denominator contains a function).

There are four types of problems you will be looking at during this section. What you select as the inside function depends on which type of problem you are faced with.

Type of problem	**Example**	**Inside function (u)**
function raised to a power	$\left(3x^4-5\right)^n$	$u=3x^4-5$
function under a radical	$\sqrt[3]{x^2+9}$	$u=x^2+9$
function is an exponent on e	e^{2x^2-x}	$u=2x^2-x$
function is in the denominator of a fraction	$\dfrac{2x-3}{x^2-3x}$	$u=x^2-3x$

Example 2: Find the indefinite integral $\int 2xe^{3x^2}\,dx$ using substitution.

Solution:

Step 1: Identify the inside function and set it equal to u.

In this example, we have a function as an exponent on *e* so the inside function would be equal to the exponent of $3x^2$.

$$u=3x^2$$

Step 2: Find the differential of *u*.

$$u=3x^2$$
$$\frac{du}{dx}=6x$$
$$du=6x\cdot dx$$

Step 3: Check if the differential of *u* is present in the integrand

In this problem you would first want to rearrange the terms in the integrand. You would want to group the "2*x*" in the integrand with *dx* since it is not part of the exponential function.

$$\int 2xe^{3x^2}dx=\int e^{3x^2}\,2x\cdot dx$$

$du = 6x\,dx$ but the integrand only contains $2x\,dx$. Therefore, you must divide *du* by 3.

Example 2 (Continued):

$$du = 6x \cdot dx$$

$$\frac{du}{3} = \frac{6x \cdot dx}{3}$$

$$\frac{1}{3}du = 2x \cdot dx$$

Step 4: Perform the substitution and find the antiderivative.

$$\begin{aligned} \int 2xe^{3x^2}dx &= \int e^{3x^2}\,2x \cdot dx \\ &= \int e^u \frac{1}{3}du \\ &= \frac{1}{3}\int e^u du \\ &= \frac{1}{3}e^u + C \end{aligned}$$

Step 5: Substitute the inside function back into the antiderivative.

$$\begin{aligned} \int 2xe^{3x^2}dx &= \frac{1}{3}e^u + C \\ &= \frac{1}{3}e^{3x^2} + C \end{aligned}$$

Example 3: Find the indefinite integral $\int \frac{\sqrt{2+\ln x}}{x}dx$ using substitution.

Solution:

Step 1: Identify the inside function and set it equal to *u*.

In this example, you have both a function under a radical and a function in the denominator of a fraction. However, the function under the radical (2 + ln *x*) is more complex than the function in the denominator (x) so the inside function would be 2 + ln *x*.

$u = 2 + \ln x$

Example 3 (Continued):

Step 2: Find the differential of *u*.

$$u = 2 + \ln x$$
$$\frac{du}{dx} = 0 + \frac{1}{x}$$
$$du = \frac{1}{x}dx$$

Step 3: Check if the differential of *u* is present in the integrand

Like in the previous example, you will first want to regroup the terms in the integrand. The inside function is under the radical, so you would factor the radical out of the fraction.

$$\int \frac{\sqrt{2+\ln x}}{x}dx = \int \sqrt{2+\ln x}\ \frac{1}{x}dx$$

In this problem the differential of *du* is present in the integrand so you do not need to make any adjustments.

Step 4: Perform the substitution and find the antiderivative.

$$\begin{aligned}\int \frac{\sqrt{2+\ln x}}{x}dx &= \int \sqrt{2+\ln x}\ \frac{1}{x}dx \\ &= \int \sqrt{u}\ \ du \\ &= \int u^{\frac{1}{2}}\ \ du \\ &= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= \frac{2}{3}u^{\frac{3}{2}} + C\end{aligned}$$

Example 3 (Continued):

Step 5: Substitute the inside function back into the antiderivative.

$$\int \frac{\sqrt{2+\ln x}}{x}\,dx = \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}\left(2+\ln x\right)^{\frac{3}{2}} + C$$