Integration by Parts

Integration by parts is a method used to reduce a complicated integral to a simpler form that could then be determined by using the integration methods that you have previous learned. This method can be used when the given integral is in the form of

$$\int u \ dv$$

where u is a function that can be reduced by differentiation and dv is a function that can be reduced by integration.

When deciding on which function should be set equal to u and dv remember that the goal of this process of integration by parts is to make the integral simpler. Once you have chosen the functions for u and dv the integral would be integrated as:

$$\int u \ dv = uv - \int v \ du$$

Example 1: Find $\int x^3 \ln x \ dx$

Solution:

Step 1: Determine u and dv

In this example we have the functions x^3 and $\ln x$. Both of these functions can easily be differentiated. However if we look at dv, it would be more difficult to integrate $\ln x$ than it would x^3, therefore you would let

$$u = \ln x \text{ and } dv = x^3 \ dx$$

Step 2: Find du and v

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$dv = x^3 dx$$

$$v = \int x^3 dx$$

$$= \frac{x^4}{4}$$

Example 1 (Continued):

Step 3: Substitute *u*, *v*, *du*, and *dv* into the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

$$\int (\ln x)\,x^3 dx = (\ln x)\left(\frac{x^4}{4}\right) - \int \left(\frac{x^4}{4}\right)\left(\frac{1}{x}\,dx\right)$$

Step 4: Simplify

$$\begin{aligned}\int (\ln x)\,x^3 dx &= (\ln x)\left(\frac{x^4}{4}\right) - \int \left(\frac{x^4}{4}\right)\left(\frac{1}{x}\,dx\right)\\ &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3\,dx\\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx\\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{x^4}{4}\right) + C\\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C\end{aligned}$$

Example 2: Find $\int \frac{(x+2)}{e^{3x}}\,dx$

Solution:

Step 1: Determine *u* and *dv*

First lets rewrite the integrand of the problem as

$$\int \frac{(x+2)}{e^{3x}}\,dx = \int (x+2)\,e^{-3x} dx$$

Now you can see that the two functions are (x + 2) and e^{-3x}. Both of these functions can easily be differentiated or integrated, so how do you choose? Recall that the derivative of e^u would be $e^u\,du$, which would not help in making the integral simpler. Therefore, you would let

$u = x + 2$ and $dv = e^{-3x}\,dx$

Example 2 (Continued):

Step 2: Find *du* and *v*

$$u = x + 2$$
$$du = dx$$

$$dv = e^{-3x}dx$$
$$v = \int e^{-3x}dx$$
$$= \frac{e^{-3x}}{-3}$$

Note: Recall that $\int e^{kx}dx = \frac{e^{kx}}{k} + C$. However, you do not need to include the constant "*C*" yet since we are only finding v.

Step 3: Substitute *u*, *v*, *du*, and *dv* into the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$
$$\int (x+2)e^{-3x}dx = (x+2)\left(\frac{e^{-3x}}{-3}\right) - \int \left(\frac{e^{-3x}}{-3}\right)(dx)$$

Step 4: Simplify

$$\int (x+2)e^{-3x}dx = (x+2)\left(\frac{e^{-3x}}{-3}\right) - \int \left(\frac{e^{-3x}}{-3}\right)(dx)$$
$$= -\frac{x+2}{3e^{3x}} - \left(-\frac{1}{3}\right)\int e^{-3x}\,dx$$
$$= -\frac{x+2}{3e^{3x}} + \frac{1}{3}\left(\frac{e^{-3x}}{-3}\right) + C$$
$$= -\frac{x+2}{3e^{3x}} - \frac{1}{9e^{3x}} + C$$

There also may be problems that you encounter which cannot be easily integrated by using integration by parts. In these cases there are tables of integrals (found in the appendix of the textbook) that can be used. These notes contain only a few of the integrals.

A sample of the Table of Integrals:

1. $\int \ln|kx|\,dx = x\ln|kx| - x + C$

2. $\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

3. $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

4. $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

5. $\int \frac{1}{x\sqrt{a^2 \pm x^2}}\,dx = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 \pm x^2}}{x}\right| + C$

Example 3: Find $\int \frac{7}{x^2 - 4}\,dx$

Solution:

Step 1: Find an integral in the table of integrals that matches this example

This example can be rewritten so that it is in the form of the integral $\int \frac{1}{x^2 - a^2}\,dx$

$$\int \frac{7}{x^2 - 4}\,dx = 7\int \frac{1}{x^2 - 2^2}\,dx$$

Step 2: Apply the selected integral to this problem

$$7\int \frac{1}{x^2 - 2^2}\,dx = 7\left[\frac{1}{2(2)}\ln\left|\frac{x-2}{x+2}\right|\right] + C$$

$$= \frac{7}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$