Law of Cosines

In the previous section, we learned how the Law of Sines could be used to solve oblique triangles in three different situations (1) where a side and two angles (SAA) were known, (2) where two angles and the included side (ASA) were known, and (3) the ambiguous case where two sides and an angle opposite one of the sides (SSA) were known. However, how would we solve oblique triangles where two sides and the included angle (SAS) or where only the three sides (SSS) are known? In these two cases, there is not enough information to use the Law of Sines so we must now use a combination of the Law of Cosines and the Law of Sines to solve the oblique triangle.

Definition of the Law of Cosines:

> If A, B, and C are the measures of the angles of an oblique triangle, and a, b, and c are the lengths of the sides opposite the corresponding angles, then the square of a side of the triangle is equal to the sum of the squares of the other two sides minus twice the product of the two sides and the cosine of the included angle.
>
> $a^2 = b^2 + c^2 - 2bc \cos A$
> $b^2 = a^2 + c^2 - 2ac \cos B$
> $c^2 = a^2 + b^2 - 2ab \cos C$

Looking at the formulas for the Law of Cosines (especially the last one) you can see that it looks almost identical to the Pythagorean Theorem except for the product at the end. But what would happen if C is 90°?

$c^2 = a^2 + b^2 - 2ab \cos C$
$c^2 = a^2 + b^2 - 2ab \cos 90°$
$c^2 = a^2 + b^2 - 2ab\ (0)$
$c^2 = a^2 + b^2$

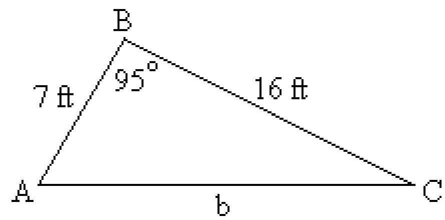
If the included angle is a right angle then the Law of Cosines is the same as the Pythagorean Theorem.

Applying the Law of Cosines:

In this first example we will look at solving an oblique triangle where the case SAS exists. For this case we will apply the following steps:

1. Use the Law of Cosines to find the side opposite to the given angle.
2. Use the Law of Sines to find the measure of the angle that is opposite of the shorter of the two given sides.
3. Use the property for the sum of interior angles of a triangle to find the remaining angle.

Example 1: Solve the given triangle rounding the lengths of the sides and the angle measures to the nearest tenth.



Solution:

Find b

$b^2 = a^2 + c^2 - 2ac \cos B$
$b^2 = (16)^2 + (7)^2 - 2(16)(7) \cos 95°$
$b^2 = 256 + 49 - 224 \cos 95°$
$b^2 \approx 305 + 19.5$
$b^2 \approx 324.5$
$b \approx 18.0$ feet

Find ∠C (the angle opposite the shorter known side)

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{7}{\sin C} \approx \frac{18}{\sin 95°}$$
$$18 \sin C \approx 7 \sin 95°$$
$$\sin C \approx \frac{7 \sin 95°}{18}$$
$$C \approx 67.2°$$

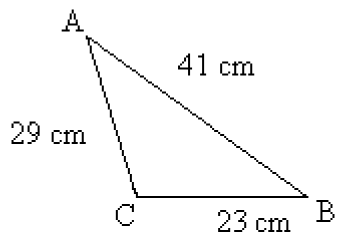
Find ∠A

$\angle A + \angle B + \angle C = 180°$
$\angle A + 95° + 67.2° \approx 180°$
$\angle A \approx 180° - 162.2°$
$\angle A \approx 17.8°$

In the next example we will look at solving an oblique triangle where the case SSS exists. For this case we will apply the following steps:

1. Use the Law of Cosines to find measure of the angle opposite of the longest side.
2. Use the Law of Sines to find the measure of one of the other two angles.
3. Use the property for the sum of interior angles of a triangle to find the remaining angle.

Example 2: Solve the given triangle rounding the lengths of the sides and the angle measures to the nearest tenth.



Solution:

Find ∠C (the angle opposite the longest side)

$c^2 = a^2 + b^2 - 2ab \cos C$
$(41)^2 = (23)^2 + (29)^2 - 2(23)(29) \cos C$
$1681 = 529 + 841 - 1334 \cos C$
$1334 \cos C = 1370 - 1681$
$1334 \cos C = -311$
$\cos C \approx -0.2331$
$C \approx 103.5°$

Find ∠A

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{23}{\sin A} \approx \frac{41}{\sin 103.5°}$$
$$41 \sin A \approx 23 \sin 103.5°$$
$$\sin A \approx \frac{23 \sin 103.5°}{41}$$
$A \approx 33.1°$

Example 2 (Continued):

Find $\angle B$

$\angle A + \angle B + \angle C = 180°$
$33.1° + \angle B + 103.5° \approx 180°$
$\angle B \approx 180° - 136.6°$
$\angle B \approx 43.4°$

Applying the Law of Cosines in a real world application:

Example 3: Two cars leave a city at the same time and travel along straight highways that differ in direction by 80. One car averages 60 miles per hour and the other averages 50 miles per hour. How far apart will the cars be after 90 minutes? Round the answer to the nearest tenth of a mile.

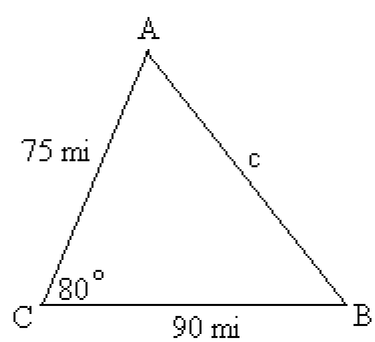
Solution:

Determine how far each car has traveled during the 90 minutes

d_1 = distance of car 1
d_2 = distance of car 2
r_1 = rate of car 1 = 60 mph
r_2 = rate of car 2 = 50 mph
t = time traveled = 90 minutes = 1.5 hours

$d_1 = r_1 * t$	$d_2 = r_2 * t$
d_1 = (60 mph) * (1.5 hrs)	d_2 = (50 mph) * (1.5 hrs)
d_1 = 90 mi	d_2 = 75 mi

Draw diagram of situation



Identify the case for this problem

This problem involves the SAS case

Example 3 (Continued):

Find c

$c^2 = a^2 + b^2 - 2ab \cos C$
$c^2 = (90)^2 + (75)^2 - 2(90)(75) \cos 80°$
$c^2 = 8100 + 5625 - 13500 \cos 80°$
$c^2 \approx 13725 - 2344.2504$
$c^2 \approx 11380.$
$c \approx 106.7$ mi

After 90 minutes the two cars would be approximately 106.7 miles apart.

Heron's Formula:

It is possible to find the area of an oblique triangle where only the lengths of the sides known without having to find the height by using Heron's Formula.

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{1}{2}(a+b+c)$$

Example 4: Find the area of a triangle with the sides of a = 7 feet, b = 11 feet, and c = 9 feet. Round the area to the nearest square foot.

Solution:

Find s (half of the perimeter)

$$s = \frac{1}{2}(a+b+c)$$
$$s = \frac{1}{2}(7+11+9)$$
$$s = 13.5$$

Find the area

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$Area = \sqrt{13.5(13.5-7)(13.5-11)(13.5-9)}$$
$$Area = \sqrt{13.5(6.5)(2.5)(4.5)}$$
$$Area = \sqrt{987.1875}$$
$$Area \approx 31.4$$

The area of the triangle is approximately 31.4 square feet.