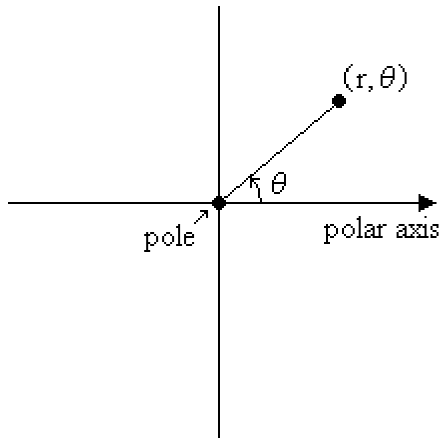
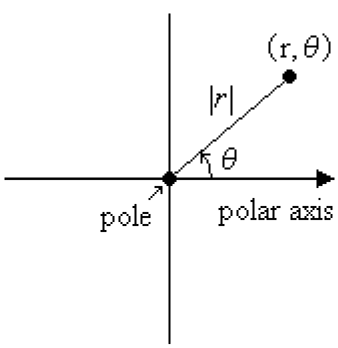
Polar Coordinates

In a rectangular coordinate system, we were plotting points based on an ordered pair of (x, y). In the polar coordinate system, the ordered pair will now be (r, θ). The ordered pair specifies a point's location based on the value of r and the angle, θ, from the polar axis.

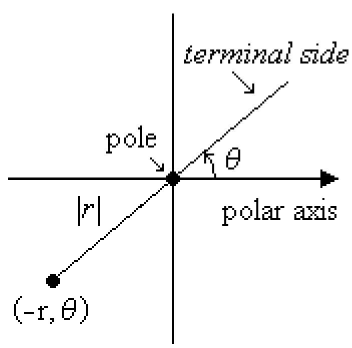


The value of r can be positive, negative, or zero. The sign of r is very important in locating the exact position of the point. The absolute value of r, |r|, is the distance between the point and the pole.

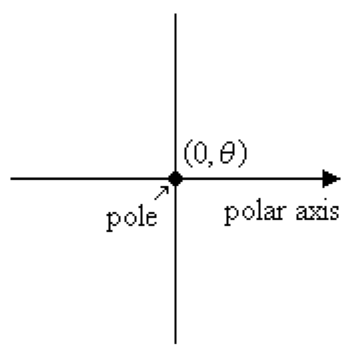
1. If r is positive (r > 0) then the point lies on the terminal side of θ
2. If r is negative (r < 0) then the point lies on the ray opposite of the terminal side of θ
3. If r is zero (r = 0) then the point lies at the pole regardless of θ



r > 0



r < 0

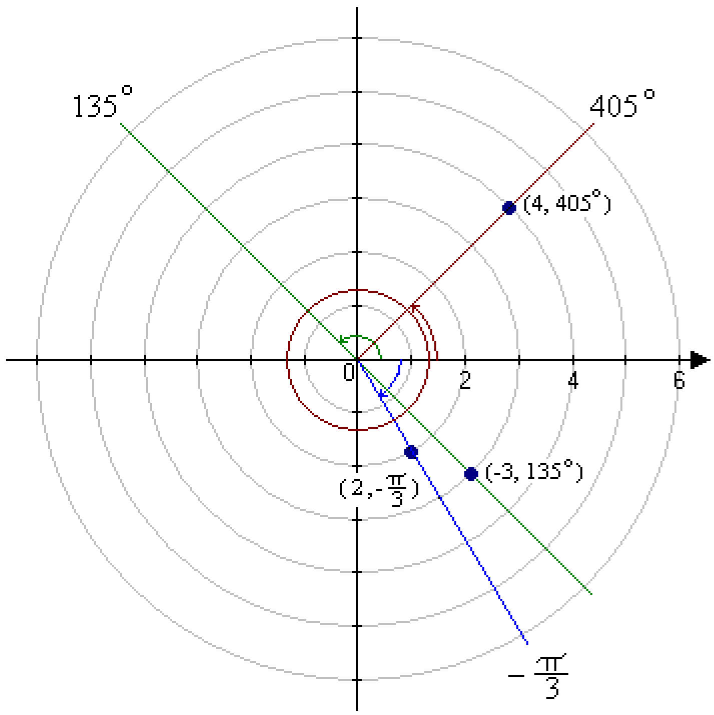


r = 0

Example 1: Plot the following polar coordinates.

a) (-3, 135°) b) $(2, -\frac{\pi}{3})$ c) (4, 405°)

Solution:



One big difference between polar and rectangular coordinates is that polar coordinates can have multiple coordinates representing the same point by adjusting the angle θ or the sign of r and the angle θ.

To find other representations for the point (r, θ):

- Add a multiple of 2π to the angle and do not change r
- Add a multiple of π to the angle and change r to –r

Example 2: Find another representation of the ordered pair $(3, \frac{\pi}{4})$ that satisfies the given conditions.

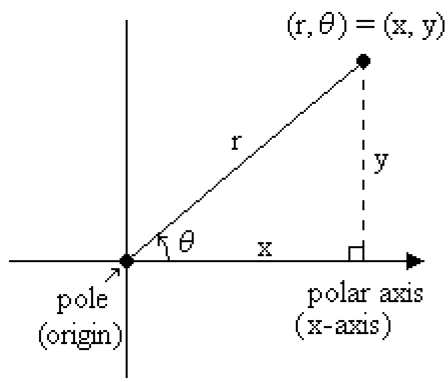
r is negative and $0 < \theta < 2\pi$

Solution:

Change r to –r and add π to the angle

$$(3, \frac{\pi}{4}) = (-3, \frac{\pi}{4} + \pi)$$
$$= (-3, \frac{5\pi}{4})$$

Since the polar axis coincides with the x-axis and the pole coincides with the origin, a relationship can be established between the polar coordinates and the rectangular coordinates for a point.



By forming a right triangle, we can use the definitions of the trigonometric functions to establish the relationship between the two sets of ordered pairs.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$
$$r\sin\theta = y \qquad r\cos\theta = x$$

The Pythagorean Theorem can also be used to establish a relationship between x, y, and r.

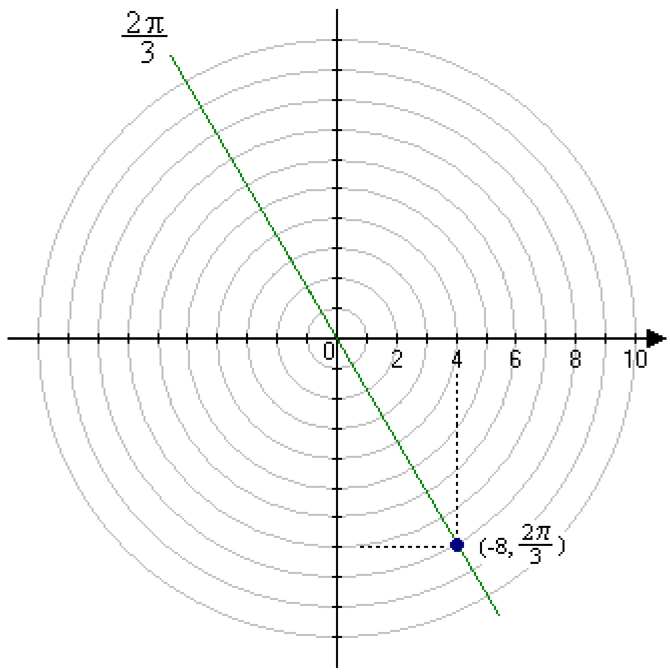
$$a^2 + b^2 = c^2$$
$$x^2 + y^2 = r^2$$

Example 3: Convert the polar coordinates $(-8, \frac{2\pi}{3})$ into rectangular coordinates.

Solution:

Plot the polar coordinates

This is not a necessary step but it will provide you with an idea of what should be the rectangular coordinates.



From the graph, we can see that the point is in the fourth quadrant making the x coordinate positive and the y coordinate negative. The rectangular coordinates would be approximately (4, -7).

Find x

$x = r \cos \theta$

$x = -8 \cos \frac{2\pi}{3}$

$x = (-8) (-\frac{1}{2})$

$x = 4$

Example 3 (Continued):

Find y

$y = r \sin \theta$

$y = -8 \sin \frac{2\pi}{3}$

$y = (-8)\left(\frac{\sqrt{3}}{2}\right)$

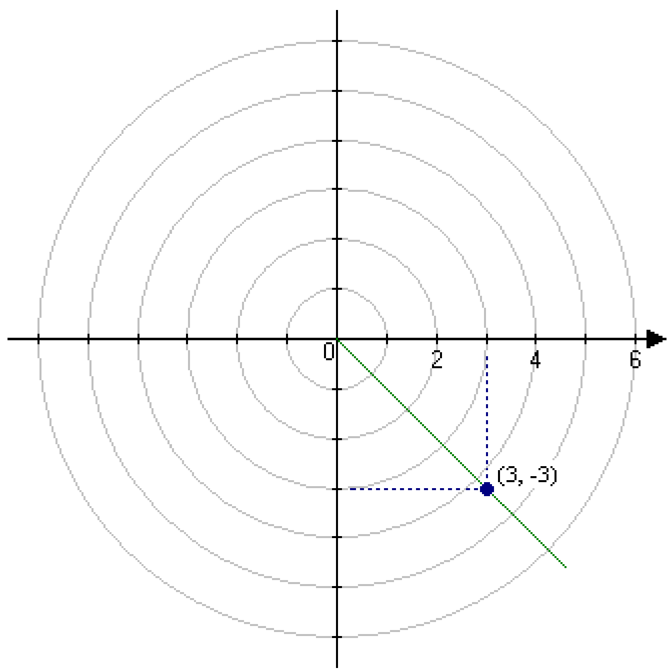
$y = -4\sqrt{3}$

The rectangular coordinates would be $(4, -4\sqrt{3})$ which is approximately (4, -7).

Example 4: Convert the rectangular coordinates (3, -3) into polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

Solution:

Plot the rectangular coordinates



From the graph, we can see that the point is in the fourth quadrant.

Example 4 (Continued):

Find r

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(3)^2 + (-3)^2}$$
$$r = \sqrt{9 + 9}$$
$$r = \sqrt{18}$$
$$r = 3\sqrt{2}$$

Find θ

$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{-3}{3}$$
$$\tan\theta = -1$$

$\tan\frac{\pi}{4} = 1$ and since the point is in quadrant IV we would subtract the reference angle for 2π to get the measurement of θ

$$\theta = 2\pi - \frac{\pi}{4}$$
$$\theta = \frac{7\pi}{4}$$

The polar coordinates would be ($3\sqrt{2}$, $\frac{7\pi}{4}$).

The relationships that have been established between the polar and rectangular coordinate systems can also be used in converting a polar equation into rectangular form or converting a rectangular equation into polar form. The next two examples will demonstrate how this is done.

Example 5: Convert the rectangular equation $x^2 + y^2 = 100$ into a polar equation that expresses r in terms of θ.

Solution:

Substitute x and y with their polar equivalents

$x = r \cos \theta$ and $y = r \sin \theta$

$x^2 + y^2 = 100$
$(r \cos \theta)^2 + (r \sin \theta)^2 = 100$

Solve for r

$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 100$	
$r^2 (\cos^2 \theta + \sin^2 \theta) = 100$	factor common term r^2
$r^2 (1) = 100$	apply Pythagorean identity $\cos^2 \theta + \sin^2 \theta = 1$
$r^2 = 100$	
$r = 10$	

Example 6: Convert the polar equation $4r \cos \theta + r \sin \theta = 8$ into a rectangular equation that expresses y in terms of x.

Solution:

Substitute r cos θ and r sin θ with their rectangular equivalents

$4r \cos \theta + r \sin \theta = 8$
$4(x) + (y) = 8$

Solve for y

$4x + y = 8$
$y = 8 – 4x$