SUM AND DIFFERENCE FORMULAS

Introduction

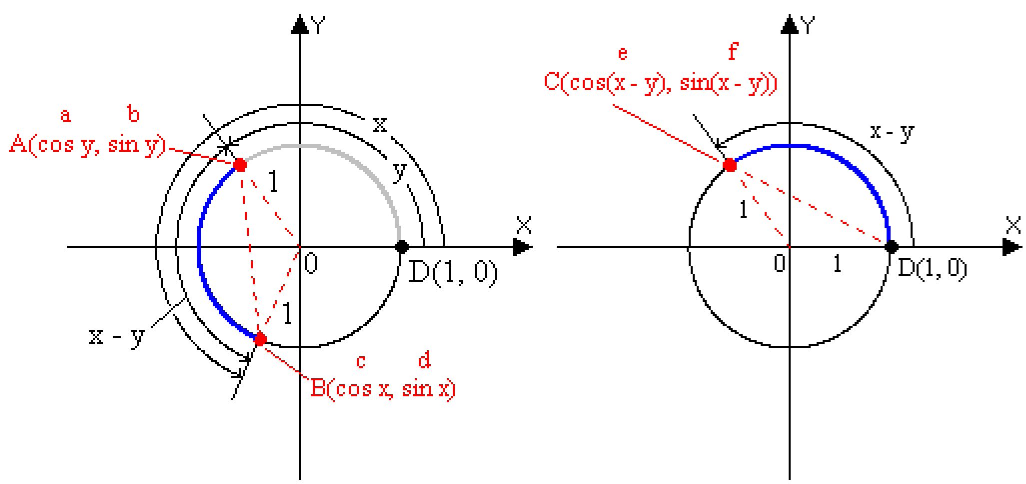
- We have several identities that we are concentrating on in this section:
    - Difference Identities for Cosine
    - Sum Identities for Cosine
    - Cofunction Identities
    - Difference Identities for Sine and Tangent
    - Sum Identities for Sine and Tangent
- Instead of just having one variable like in the basic identities, two variables are involved in the identities of this section.

Difference Identities for Cosine

Equation No. 1: $\cos(x - y) = (\cos x)(\cos y) + (\sin x)(\sin y)$

This is the difference identity for cosine

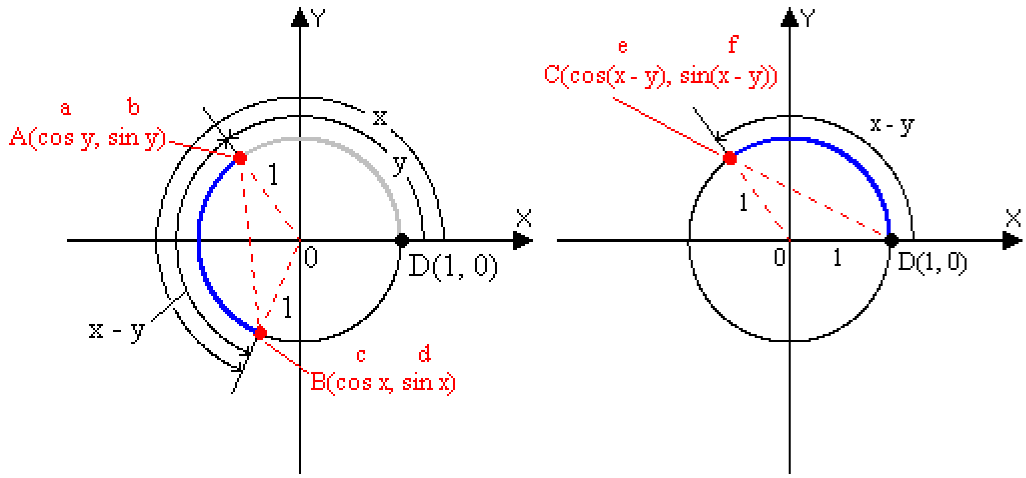
- To prove the equation above, the unit circle below assumes that x and y are within the interval $(0, 2\pi)$ and $x > y > 0$. All real numbers and angles in radian or in degrees are represented by periodicity and basic identities.



- The angles and arcs on the unit circle are to associate x and y.

- Labeling the points are done by using the definitions of the trigonometric functions (sine, cosine, tangent, cotangent, cosecant, and secant), in this case only sine and cosine are utilized.

Establishing the Difference Identity for Cosine



- If the triangle *AOB*, rotates clockwise about the origin until point A coexists with point D, then point B will be where point C is located on the unit circle. Since the rotations retains lengths:

  d (A, B) = d (C, D)

  $$\sqrt{(c-a)^2+(d-b)^2}=\sqrt{(1-e)^2+(0-f)^2}$$
  $$(c-a)^2+(d-b)^2=(1-e)^2+f^2$$
  $$c^2-2ac+a^2+d^2-2db+b^2=1-2e+e^2+f^2$$

Equation No. 2: $(c^2+d^2)+(a^2+b^2)-2ac-2db=1-2e+(e^2+f^2)$

- Considering the fact that points A, B, and C are on the unit circle:

  $c^2+d^2 = 1$,
  $a^2+b^2 = 1$, and
  $e^2+f^2 = 1$

  equation no. 2, boils down to:

  $e = ac + bd$

- Ok, I know what you are thinking, all of these letters and what does it have to do with cos (x – y) = (cos y)(cos x) + (sin y)(sin x)
- Well, let's put the equations together now by replacing *e, a, c, b*, and *d* with cos (x – y), cos y, cos x, sin y, and sin x, respectively:

Establishing Continues…

$e = ac + bd$
$\rightarrow$ cos (x – y) = (cos y)(cos x) + (sin y)(sin x)
$\rightarrow$ **cos (x – y) = (cos x)(cos y) + (sin x)(sin y)**

- Now, we have established the **difference identity for cosine**.

Sum Identity for Cosine

- In order to achieve the sum identity, we replace y with –y on the difference equation (cos x)(cos y) + (sin x)(sin y).
- Then, use the identities for negatives [cos (-y) = cos y; sin (-y) = -sin y], we obtain:

cos (x + y) = (cos x)(cos y) – (sin x)(sin y)
↑
The **Sum Identity for Cosine**

- *Note:* cos (x – y) ≠ cos x – cos y
  cos (x + y) ≠ cos x + cos y

Cofunction Identities

- From the difference identity for cosine equation, we are going to attain the cofunction identities for cosine, sine, and tangent.
- First we take the difference identity:

cos (x – y) = (cos x)(cos y) + (sin x)(sin y)

- Let x = π/2:

cos (π/2 – y) = (cos π/2)(cos y) + (sin π/2)(sin y)

- Then, we find that cos π/2 = 0, and sin π/2 =1, according to the Unit Circle, therefore:

cos (π/2 – y) = (0) cos y + (1) sin y
= sin y
cos (π/2 – y) = sin y
↑
We have established the **Cofunction identity for Cosine**.

- For y any real number or radian measure. Replace π/2 with 90 degrees, if y is measured in degrees.

Cofunction Identities, Continued…

- We have the cosine taken care of, now let us do the cofunction identity for sine.
- First we take the cofunction identity for cosine:

$$\cos(\pi/2 - y) = \sin y$$

- Then, we let $y = \pi/2 - x$:

$$\cos(\pi/2 - (\pi/2 - x)) = \sin(\pi/2 - x)$$
$$\cos(0 + x) = \sin(\pi/2 - x)$$
$$\cos x = \sin(\pi/2 - x)$$
$$\sin(\pi/2 - x) = \cos x$$
↑
Cofunction identity for sine

- For any real number x or radian measure. Replace π/2 with 90 degrees if x is in degree measure.

Cofunction Identities Conclusion…

- **The cofunction for tangent** is:

$$\mathbf{\tan(\pi/2 - x) = \cot x}$$

- Where x is any real number or radian measure. Replace π/2 with 90 degrees, if x is in degree measure.
- To conclude:
  - cosine, cotangent, and cosecant are basically cofunctions of sine, tangent, and secant, respectively.
  - Basically, cosine, cotangent, and cosecant means, complements sine, tangent, and secant, respectively.
  - When, $0 < x < 90$ degrees, then x and $90 - x$ are complementary angles.
- Now that we have the cofunction identities in place, we can now move on to the sum and difference identities for sine and tangent.

Difference Identity for Sine

- To arrive at the difference identity for sine, we use 4 verified equations and some algebra:
  - cofunction identity for cosine equation
  - difference identity for cosine equation
  - cofunction identity for sine equation
  - the identities for negatives

Difference Identity for Sine (Continued…)

- First, use the cofunction identity for cosine :

  cos (π/2 – y) = sin y
  ↓
  sin (x - y) = cos [π/2 – (x - y)]

- Second, we apply algebra:

  sin (x - y) = cos [(π/2 – x) – (-y)]

- Third, use the difference identity for cosine equation:

  cos (x – y) = (*cos x*)(*cos y*) + (*sin x*)(*sin y*)
  ↓
  cos (x – y) = cos (π/2 – x) cos (-y) + sin (π/2 – x) sin (-y)

- Finally, use cofunction identity for cosine and sine, also the identities for negatives:

  cos (π/2 – y) = sin y ; sin (π/2 – x) = cos x ; sin (-x) = -sin x; cos (-x) = cos x
  ↓
  sin (x – y) = (sin x)(cos y) – (cos x)(sin y) → Difference Identity for Sine

Sum Identity for Sine

- To obtain the sum identity for sine, we replace y with –y in the difference identity for cosine equation, as follows:

  sin (x – (-y)) = (sin x)(cos (-y)) – (cos x)(sin (-y))
  ↓
  sin (x + y) = (sin x)(cos y) + (cos x)(sin y)
  (identities for negatives was utilized to derive the sum identity for sine equation)

Difference & Sum Identity for Tangent

- To attain the difference identity for tangent, we use both sine and cosine difference identities:

$$\tan(x-y) = \frac{\sin(x-y)}{\cos(x-y)}$$
$$= \frac{(\sin x)(\cos y)-(\cos x)(\sin y)}{(\cos x)(\cos y)+(\sin x)(\sin y)}$$

- take the numerator and denominator and divide it by cos x cos y

$$\tan(x-y)=\frac{(\sin x)(\cos y)-(\cos x)(\sin y)}{(\cos x)(\cos y)+(\sin x)(\sin y)}$$

$$=\frac{\dfrac{(\sin x)(\cos y)-(\cos x)(\sin y)}{(\cos x)(\cos y)}}{\dfrac{(\cos x)(\cos y)+(\sin x)(\sin y)}{(\cos x)(\cos y)}}$$

- algebraic operations are applied

$$\tan(x-y)=\frac{\dfrac{(\sin x)(\cos y)-(\cos x)(\sin y)}{(\cos x)(\cos y)}}{\dfrac{(\cos x)(\cos y)+(\sin x)(\sin y)}{(\cos x)(\cos y)}}$$

$$=\frac{\dfrac{(\sin x-\sin y)}{(\cos x)(\cos y)}}{1+\dfrac{(\sin x)(\sin y)}{(\cos x)(\cos y)}}$$

$\downarrow$

$$\tan(x-y)=\frac{\dfrac{(\sin x-\sin y)}{(\cos x)(\cos y)}}{1+\dfrac{(\sin x)(\sin y)}{(\cos x)(\cos y)}}$$

$$=\frac{\tan x-\tan y}{1+(\tan x)(\tan y)}$$

Difference Identity for Tangent

- To obtain the sum identity, we replace every y with –y in the equation above:

$$\tan(x-y)=\frac{\tan x-\tan y}{1+(\tan x)(\tan y)}$$

$$\tan(x-(-y))=\frac{\tan x-\tan(-y)}{1+(\tan x)(\tan(-y))}$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-(\tan x)(\tan y)}$$

Sum Identity for Tangent

by Shavana Gonzalez

Example 1 (Difference Identity): Simplify $\sin(x - \pi/2)$

Answer

- Let's simplify $\sin(x - \pi/2)$, using the difference identity, $\sin(x - y) = (\sin x)(\cos y) - (\cos x)(\sin y)$

$$\begin{aligned} \sin(x - \pi/2) &= (\sin x)(\cos \pi/2) - (\cos x)(\sin \pi/2) \\ &= (\sin x)(0) - (\cos x)(1) \\ &= \mathbf{-\cos x} \end{aligned}$$

Example 2 (Sum Identity): Simplify $\cos(\pi + x)$

Answer

- Using the sum identity, $\cos(x + y) = (\cos x)(\cos y) - (\sin x)(\sin y)$:

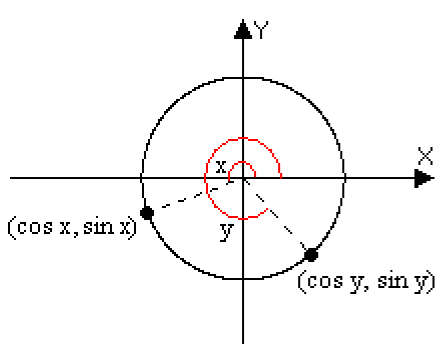
$$\begin{aligned} \cos(\pi + x) &= (\cos \pi)(\cos x) - (\sin \pi)(\sin x) \\ &= (-1)(\cos x) - (0)(\sin x) \\ &\mathbf{= -\cos x} \end{aligned}$$

Example 3 (Finding Exact Values): If $\sin x = -1/3$, and $\cos y = 2/3$, find the exact value of $\cos(x + y)$. The number x is an angle in quadrant III, and y is quadrant IV. Don't use a calculator.

Answer

$$\cos(x + y) = (\cos x)(\cos y) - (\sin x)(\sin y)$$

- We already know sin x and cos y, however we don't know cos x and sin y. To find cos x and sin y, let's do the Unit Circle method.
- $\sin x = -1/3$ and x is in quadrant III, let's find cos x:



Example 3 (Continued):

$\cos x = a$
$a^2 + b^2 = 1$
$a^2 + (-1/3)^2 = 1$
$a^2 = 1 - 1/9$

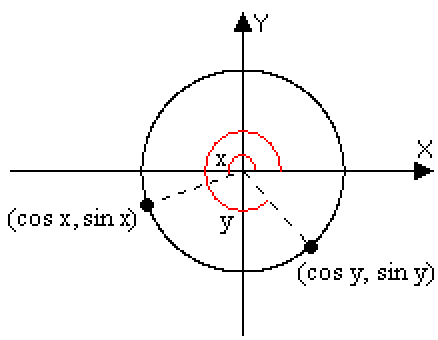
$$\sqrt{a^2} = \sqrt{\frac{8}{9}}$$

$$a = \pm\frac{2\sqrt{2}}{3}$$

- Due to the fact that x is in quadrant III, the result is negative:

$$a = -\frac{2\sqrt{2}}{3} \rightarrow \cos x = -\frac{2\sqrt{2}}{3}$$

- $\cos y = 2/3$, and y is in quadrant IV, let's find sin y:



$\sin y = b$
$a^2 + b^2 = 1$
$(2/3)^2 + b^2 = 1$
$4/9 + b^2 = 1$
$b^2 = 1 - 4/9$
$b^2 = 5/9$

$$b = \pm\frac{\sqrt{5}}{3}$$

- Since, y is in quadrant IV, the answer is negative:

$$b = -\frac{\sqrt{5}}{3}$$

$$\sin y = -\frac{\sqrt{5}}{3}$$

Example 3 (Continued):

- Now, that we have sin x, sin y, cos x, and cos y, we can now solve cos (x + y):

$$\begin{aligned}\cos(x+y) &= (\cos x)(\cos y) - (\sin x)(\sin y)\\ &= \left(-\frac{2\sqrt{2}}{3}\right)\left(\frac{2}{3}\right) - \left(-\frac{1}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)\\ &= \left(-\frac{4\sqrt{2}}{9}\right) - \left(\frac{\sqrt{5}}{9}\right)\\ &= -\frac{4\sqrt{2}+\sqrt{5}}{9}\end{aligned}$$

- The exact value is $-\frac{4\sqrt{2}+\sqrt{5}}{9}$

Example 4: Establish the identity of $\tan x - \tan y = \frac{\sin(x-y)}{(\cos x)(\cos y)}$

Answer

$$\tan x - \tan y = \frac{\sin(x-y)}{(\cos x)(\cos y)}$$

$$\tan x - \tan y = \frac{(\sin x)(\cos y) - (\cos x)(\sin y)}{(\cos x)(\cos y)} \qquad \text{(difference identity)}$$

$$\tan x - \tan y = \frac{(\sin x)(\cos y)}{(\cos x)(\cos y)} - \frac{(\cos x)(\sin y)}{(\cos x)(\cos y)} \qquad \text{(algebra)}$$

$$\tan x - \tan y = \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y} \qquad \text{(quotient identities)}$$

$$\tan x - \tan y = \tan x - \tan y$$