Trigonometric Equations

Just as we can have polynomial, rational, exponential, or logarithmic equation, for example, we can also have trigonometric equations that must be solved. A trigonometric equation is one that contains a trigonometric function with a variable. For example, $\sin x + 2 = 1$ is an example of a trigonometric equation. The equations can be something as simple as this or more complex like $\sin^2 x - 2\cos x - 2 = 0$. The steps taken to solve the equation will depend on the form in which it is written and whether we are looking to find all of the solutions or just those within a specified interval such as $[0, 2\pi)$.

Solving for all solutions of a trigonometric equation

Back when we were solving for theta, θ, using the inverse trigonometric function we were limiting the interval for θ depending on the trigonometric function. For example, θ was limited to the interval of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function. However, when we are solving a trigonometric equation for all of the solutions we will not limit the interval and must adjust the values to take into account the periodic nature of the trigonometric function. The functions sine, cosine, secant, and cosecant all have a period of 2π so we must add the term $2n\pi$ to include all of the solutions. Tangent and cotangent have a period of π so for these two functions the term $n\pi$ would be added to obtain all of the solutions.

Example 1: Find all of the solutions for the equation $2\cos x = \sqrt{2}$.

Solution:

Isolate the function on one side of the equation

$$2\cos x = \sqrt{2}$$
$$\cos x = \frac{\sqrt{2}}{2}$$

Identify the quadrants for the solutions on the interval $[0, 2\pi)$

Cosine is positive in quadrants I and IV

Solve for the variable

$$x = \frac{\pi}{4} \text{ (quadrant I)} \qquad x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} \text{ (quadrant IV)}$$

Example 1 (Continued):

Add $2n\pi$ to the values of x

$$x = \frac{\pi}{4} + 2n\pi \qquad \text{and} \qquad x = \frac{7\pi}{4} + 2n\pi$$

Example 2: Find all of the solutions for the equation $\tan x = \sqrt{3}$.

Solution:

Identify the quadrants for the solutions on the interval $[0, \pi)$

Note: On this problem we are using the interval $[0, \pi)$ instead of $[0, 2\pi)$ because tangent has a period of π.

Tangent is positive in quadrants I

Solve for the variable

$$x = \frac{\pi}{3}$$

Add $n\pi$ to the value of x

$$x = \frac{\pi}{3} + n\pi$$

Solving trigonometric equations with a multiple angle

The trigonometric equations to be solved will not always have just "x" as the angle. There will be times where you will have angles such as 3x or $\frac{x}{2}$. For equations like this, you will begin by solving the equation for all of the possible solutions by adding $2n\pi$ or $n\pi$ (depending on the trigonometric function involved) to values. You would then substitute values in for n starting at 0 and continuing until all of the values within the specified interval have been found.

Example 3: Solve the equation csc $2x = -1$ on the interval $[0, 2\pi)$.

Solution:

Identify the quadrants for the solutions on the interval $[0, 2\pi)$

Cosecant is negative in quadrants III and IV

Solve for the angle 2x

Cosecant is equal to -1 only at $\frac{3\pi}{2}$ therefore

$$2x = \frac{3\pi}{2}$$

Add $2n\pi$ to the angle and solve for x

$$2x = \frac{3\pi}{2} + 2n\pi$$

$$\tfrac{1}{2}(2x) = \tfrac{1}{2}\left(\frac{3\pi}{2} + 2n\pi\right)$$

$$x = \frac{3\pi}{4} + n\pi$$

Now substitute values in for n starting with 0 until the angle is outside of the interval $[0, 2\pi)$

$n = 0$

$$x = \frac{3\pi}{4} + (0)\pi$$

$$x = \frac{3\pi}{4}$$

$n = 1$

$$x = \frac{3\pi}{4} + (1)\pi$$

$$x = \frac{3\pi}{4} + \frac{4\pi}{4}$$

$$x = \frac{7\pi}{4}$$

$n = 2$

$$x = \frac{3\pi}{4} + (2)\pi$$

This value will exceed 2π so it cannot be a solution

The only solutions to the equation are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$

Example 4: Solve the equation 2 cos 4x = -1 on the interval [0, 2π).

Solution:

Isolate the function on one side of the equation

$$2\cos 4x = -1$$
$$\cos 4x = -\frac{1}{2}$$

Identify the quadrants for the solutions on the interval [0, 2π)

Cosine is negative in quadrants II and III

Solve for the angle 4x

Cosine is equal to $\frac{1}{2}$ at $\frac{\pi}{3}$ so the angles in quadrants II and III are

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ (quadrant II)} \quad \text{and} \quad \pi + \frac{\pi}{3} = \frac{4\pi}{3} \text{ (quadrant III)}$$

$$4x = \frac{2\pi}{3} \quad \text{and} \quad 4x = \frac{4\pi}{3}$$

Add 2nπ to the angle and solve for x

$$4x = \frac{2\pi}{3} + 2n\pi \quad \text{and} \quad 4x = \frac{4\pi}{3} + 2n\pi$$

$$\tfrac{1}{4}(4x) = \tfrac{1}{4}\left(\frac{2\pi}{3} + 2n\pi\right) \quad \text{and} \quad \tfrac{1}{4}(4x) = \tfrac{1}{4}\left(\frac{4\pi}{3} + 2n\pi\right)$$

$$x = \frac{\pi}{6} + \frac{n\pi}{2} \quad \text{and} \quad x = \frac{\pi}{3} + \frac{n\pi}{2}$$

Now substitute values in for n starting with 0 until the angle is outside of the interval [0, 2π)

n = 0

$$x = \frac{\pi}{6} + \frac{(0)\pi}{2} \qquad x = \frac{\pi}{3} + \frac{(0)\pi}{2}$$

$$x = \frac{\pi}{6} \qquad x = \frac{\pi}{3}$$

Example 4 (Continued):

n = 1

$$x = \frac{\pi}{6} + \frac{(1)\pi}{2} \qquad x = \frac{\pi}{3} + \frac{(1)\pi}{2}$$

$$x = \frac{\pi}{6} + \frac{3\pi}{6} \qquad x = \frac{2\pi}{6} + \frac{3\pi}{6}$$

$$x = \frac{4\pi}{6} \qquad x = \frac{5\pi}{6}$$

$$x = \frac{2\pi}{3}$$

n = 2

$$x = \frac{\pi}{6} + \frac{(2)\pi}{2} \qquad x = \frac{\pi}{3} + \frac{(2)\pi}{2}$$

$$x = \frac{\pi}{6} + \frac{6\pi}{6} \qquad x = \frac{\pi}{3} + \frac{3\pi}{3}$$

$$x = \frac{7\pi}{6} \qquad x = \frac{4\pi}{3}$$

n = 3

$$x = \frac{\pi}{6} + \frac{(3)\pi}{2} \qquad x = \frac{\pi}{3} + \frac{(3)\pi}{2}$$

$$x = \frac{\pi}{6} + \frac{9\pi}{6} \qquad x = \frac{2\pi}{6} + \frac{9\pi}{6}$$

$$x = \frac{10\pi}{6} \qquad x = \frac{11\pi}{6}$$

$$x = \frac{5\pi}{3}$$

n = 4

$$x = \frac{\pi}{6} + \frac{(4)\pi}{2} \qquad x = \frac{\pi}{3} + \frac{(4)\pi}{2}$$

$$x = \frac{\pi}{6} + 2\pi \qquad x = \frac{\pi}{3} + 2\pi$$

If n = 4 then this will add 2π to the angles and put them outside of the restricted interval.

Therefore, the solutions are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}$, and $\frac{11\pi}{6}$.

Trigonometric equations in quadratic form

The trigonometric equations can also be written in the quadratic form of $au^2 + bu + c = 0$ where u is a trigonometric function. The methods that can be used to solve these equations are the same as those used when solving quadratic equations – factoring, square root property, completing the square, and the quadratic formula. The method that you use will depend on the values of a, b, and c. If the equation can be factored then this would be your first option.

Example 5: Solve the following trigonometric equation in quadratic form on the interval $[0, 2\pi)$.

$$\cos^2 x - 2 \cos x = 3$$

Solution:

Group all terms on the left side so that it is equal to 0

$$\cos^2 x - 2 \cos x - 3 = 0$$

Let u represent the trigonometric function cos x

$$u = \cos x$$

$$(\cos x)^2 - 2 \cos x - 3 = 0$$
$$u^2 - 2u - 3 = 0$$

Factor the quadratic equation

$$(u + 1)(u - 3) = 0$$

Solve for u

$u + 1 = 0$ or $u - 3 = 0$
$u = -1$ or $u = 3$

Substitute the cosine function back in for u

$\cos x = -1$ or $\cos x = 3$

cos x cannot be greater than 1 so $\cos x = 3$ has no solutions

Solve for x

$$x = \pi$$

Example 6: Solve the following trigonometric equation in quadratic form on the interval $[0, 2\pi)$.

$$\tan^2 x - 2 = 3 \tan x$$

Solution:

Group all terms on the left side so that it is equal to 0

$$\tan^2 x - 3 \tan x - 2 = 0$$

Let u represent the trigonometric function tan x

$$u = \tan x$$

$$(\tan x)^2 - 3 \tan x - 2 = 0$$
$$u^2 - 3u - 2 = 0$$

Factor the quadratic equation

The equation cannot be factored so the quadratic formula must be used

Solve for u

$a = 1$, $b = -3$, and $c = -2$

$$u = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)}$$

$$u = \frac{3 \pm \sqrt{9+8}}{2}$$

$$u = \frac{3 \pm \sqrt{17}}{2}$$

$$u = \frac{3 - \sqrt{17}}{2} \quad \text{or} \quad u = \frac{3 + \sqrt{17}}{2}$$

Substitute the tangent function back in for u

$$\tan x = \frac{3 - \sqrt{17}}{2} \quad \text{or} \quad \tan x = \frac{3 + \sqrt{17}}{2}$$

$$\tan x \approx -0.5616 \quad \text{or} \quad \tan x \approx 3.5616$$

Example 6 (Continued):

Solve for the reference angle θ

$\theta \approx \tan^{-1}(0.5616)$ or $\theta \approx \tan^{-1}(3.5616)$

$\theta \approx 0.5117$ or $\theta \approx 1.2971$

Solve for the values of x within the interval $[0, 2\pi)$

$\tan x \approx -0.5616$

tan x is negative in quadrants II and IV

$x \approx \pi - 0.5117$ or $x \approx 2\pi - 0.5117$

$x \approx 2.6299$ or $x \approx 5.7715$

$\tan x \approx 3.5616$

tan x is positive in quadrants I and III

$x \approx 1.2971$ or $x \approx \pi + 1.2971$

$x \approx 4.4387$

The solutions to the equation (rounded to four decimal places) are 1.2971, 2.6299, 4.4387, and 5.7715.

Using identities to solve trigonometric equations

There could also be equations where two or more trigonometric functions are contained within the equation. If the functions can be separated by factoring the equation then you can solve the equation using the factoring method. However, if it is not possible to factor the equation then you must use the different trigonometric identities to rewrite the function in a single trigonometric function or in a form that can be solved by factoring.

Example 7: Use trigonometric identities to solve the following equation on the interval $[0, 2\pi)$.

$$2 \sin^2 x + \cos x = 1$$

Solution:

Use the Pythagorean identity $\sin^2 x = 1 - \cos^2 x$ to replace $\sin^2 x$ in the equation

$$2 \sin^2 x + \cos x = 1$$
$$2 (1 - \cos^2 x) + \cos x = 1$$
$$2 - 2 \cos^2 x + \cos x = 1$$

Example 7 (Continued):

Group all terms on the left side so that it is equal to 0

$2 - 2\cos^2 x + \cos x - 1 = 0$
$-2\cos^2 x + \cos x + 1 = 0$

Multiply the equation by –1 to make the leading coefficient positive

$-1(-2\cos^2 x + \cos x + 1 = 0)$
$2\cos^2 x - \cos x - 1 = 0$

Let u represent the trigonometric function cos x

$u = \cos x$

$2(\cos x)^2 - \cos x - 1 = 0$
$2u^2 - u - 1 = 0$

Factor the quadratic equation

$(2u + 1)(u - 1) = 0$

Solve for u

$2u + 1 = 0$ or $u - 1 = 0$
$2u = -1$ or $u = 1$
$u = -\frac{1}{2}$

Substitute the cosine function back in for u

$\cos x = -\frac{1}{2}$ or $\cos x = 1$

Identify the quadrants for the solutions on the interval $[0, 2\pi)$

Cosine is negative in quadrants II and III and is 1 at 0

Example 7 (Continued):

Solve for x

$\cos x = -\frac{1}{2}$

Cosine is equal to $\frac{1}{2}$ at $\frac{\pi}{3}$ so the angles in quadrants II and III are

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (quadrant II) and $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ (quadrant III)

$x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$

$\cos x = 1$

$x = 0$

Add $2n\pi$ to the angle and solve for x

$x = \frac{2\pi}{3} + 2n\pi$ $\quad x = \frac{4\pi}{3} + 2n\pi$ $\quad x = 0 + 2n\pi$

Now substitute values in for n starting with 0 until the angle is outside of the interval $[0, 2\pi)$

$n = 0$

$x = \frac{2\pi}{3} + 2(0)\pi$ $\quad x = \frac{4\pi}{3} + 2(0)\pi$ $\quad x = 0 + 2(0)\pi$

$x = \frac{2\pi}{3}$ $\quad x = \frac{4\pi}{3}$ $\quad x = 0$

$n = 1$

$x = \frac{2\pi}{3} + 2(1)\pi$ $\quad x = \frac{4\pi}{3} + 2(1)\pi$ $\quad x = 0 + 2(1)\pi$

When $n = 1$ we will be adding 2π to the angles which will put them outside of the interval $[0, 2\pi)$.

So the solutions for the equation are 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

Example 8: Use trigonometric identities to solve the following equation on the interval $[0, 2\pi)$.

$$\tan x + \sec^2 x = 3$$

Solution:

Use the Pythagorean identity $\sec^2 x = 1 + \tan^2 x$ to replace $\sec^2 x$ in the equation

$$\tan x + \sec^2 x = 3$$
$$\tan x + (1 + \tan^2 x) = 3$$
$$\tan x + 1 + \tan^2 x = 3$$

Group all terms on the left side so that it is equal to 0

$$\tan^2 x + \tan x + 1 - 3 = 0$$
$$\tan^2 x + \tan x - 2 = 0$$

Let u represent the trigonometric function tan x

$$u = \tan x$$

$$(\tan x)^2 + \tan x - 2 = 0$$
$$u^2 + u - 2 = 0$$

Factor the quadratic equation

$$(u + 2)(u - 1) = 0$$

Solve for u

$$u + 2 = 0 \quad \text{or} \quad u - 1 = 0$$
$$u = -2 \quad \text{or} \quad u = 1$$

Substitute the tangent function back in for u

$$\tan x = -2 \quad \text{or} \quad \tan x = 1$$

Solve for the reference angle θ

$$\theta = \tan^{-1}(2) \quad \text{or} \quad \theta = \tan^{-1}(1)$$
$$\theta \approx 1.1071 \quad \text{or} \quad \theta = \frac{\pi}{4}$$

Example 8 (Continued):

Solve for the values of x within the interval $[0, 2\pi)$

$\tan x = -2$

$\tan x$ is negative in quadrants II and IV

$x \approx \pi - 1.1071$ or $x \approx 2\pi - 1.1071$

$x \approx 2.0345$ or $x \approx 5.1761$

$\tan x = 1$

$\tan x$ is positive in quadrants I and III

$x = \frac{\pi}{4}$ or $x = \pi + \frac{\pi}{4}$

$x = \frac{5\pi}{4}$

The solutions to the equation are 2.0345 and 5.1761 (rounded to four decimal places), $\frac{\pi}{4}$, and $\frac{5\pi}{4}$.