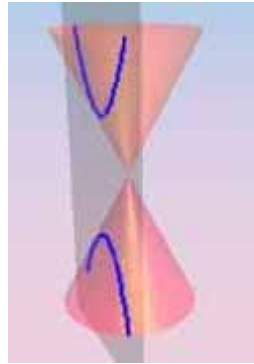


HYPERBOLA

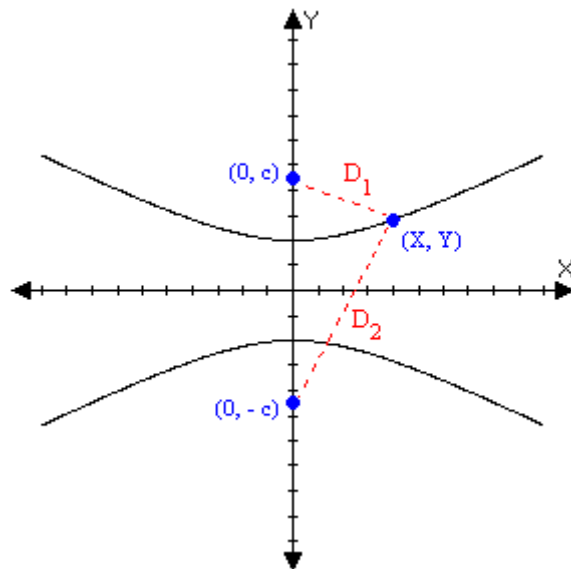
This section will discuss the conic section known as a hyperbola. A hyperbola is said to be created when a plane cuts completely through both nappes (as seen in the following picture) but not through the vertex.



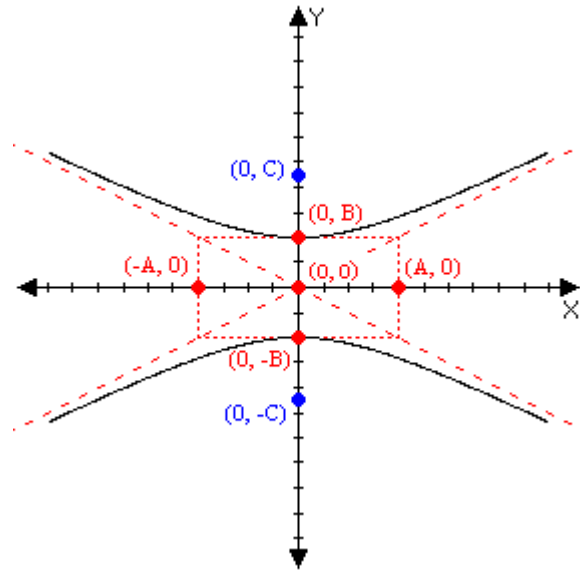
and is defined as:

a set of all points the difference of whose distances from distinct fixed points (foci) are constant.

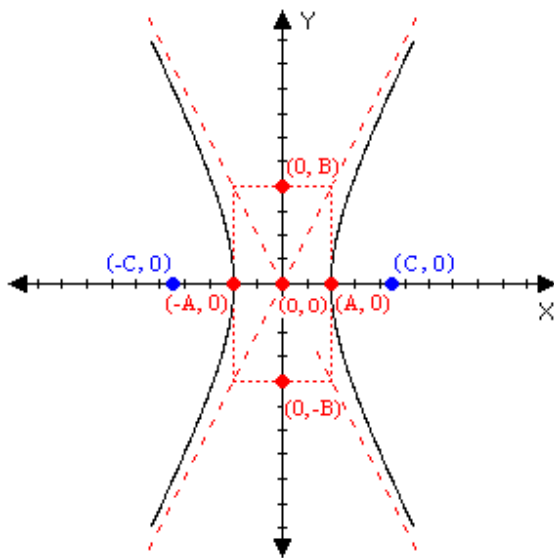
The picture below illustrates this.



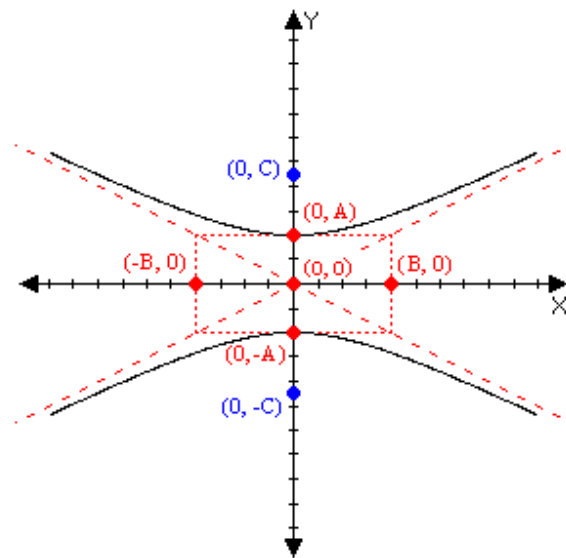
A line that passes through both foci and intersects the hyperbola at the vertex of each branch is known as the transverse axis. The midpoint of this chord is known as the center. An aid that is used in the sketching of a hyperbola is the determination of its asymptotes. Each hyperbola will have two asymptotes that intersect at the center of the hyperbola. A further graphing aid is the construction of a rectangle.



The dimensions of the rectangle are $2a$ and $2b$. The line segment of length $2b$ joining either $(0, b)$ and $(0, -b)$ or $(b, 0)$ and $(-b, 0)$, depending on the direction of the transverse axis, is referred to as the conjugate axis. Note that the transverse axis may be horizontal or vertical depending on the location of the foci. The illustration below represents this idea.



Horizontal Transverse Axis



Vertical Transverse Axis

Because the transverse axis may be horizontal or vertical, the standard form of the equation of a hyperbola is made up of two forms when the center is at the origin.

Transverse axis horizontal

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Transverse axis vertical

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

The foci for a hyperbola is found by using the formula:

$$c^2 = a^2 + b^2$$

The lengths of the axis are given by:

Transverse axis

$$2a$$

Conjugate axis

$$2b$$

The vertices for a hyperbola are:

Horizontal transverse axis

$$(a, 0) \text{ and } (-a, 0)$$

Vertical transverse axis

$$(0, a) \text{ and } (0, -a)$$

The asymptotes of the hyperbola are:

Horizontal transverse axis

$$y = \pm \frac{b}{a} x$$

Vertical transverse axis

$$y = \pm \frac{a}{b} x$$

The following examples will show how to use these formulas when solving for problems involving hyperbolas.

Example 1: Find the standard form of the equation of the hyperbola with foci at $(-3, 0)$ and $(3, 0)$ and vertices at $(-2, 0)$ and $(2, 0)$ with the center at the origin.

Solution:

Step 1: Analysis.

- a) Since the foci and vertices are in the form $(\pm a, 0)$ the transversal axis is horizontal. This means the standard equation is in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

- b) Since the foci are given as $(\pm 3, 0)$, $c = 3$.

- c) Since the vertices are given as $(\pm 2, 0)$, $a = 2$

Example 1 (Continued):

Step 2: Solve for b .

Using the foci formula b is solved for:

$$c^2 = a^2 + b^2$$

$$c^2 - a^2 = b^2$$

$$(3)^2 - (2)^2 = b^2$$

$$9 - 4 = b^2 = 5$$

$$b = \sqrt{5}$$

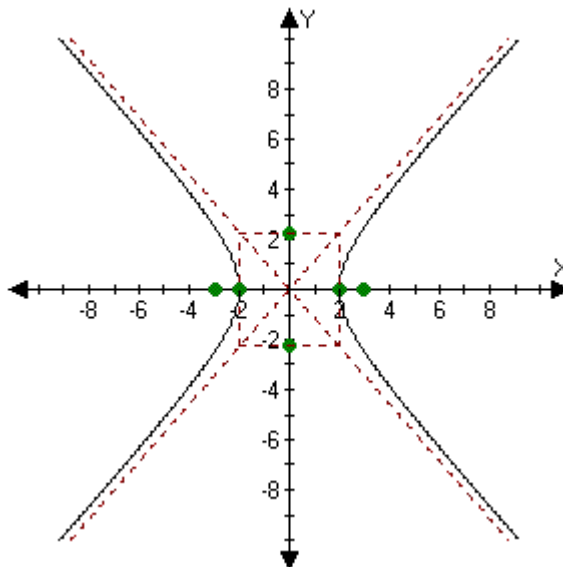
Step 3: Substitute and solve.

The values found in step 1 and 2 are substituted into the formula to yield the equation of the hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{(2)^2} - \frac{y^2}{(\sqrt{5})^2} = 1$$

Step 4: Plot and graph.



Example 2: Sketch the graph of the hyperbola whose equation is $4x^2 - y^2 = 16$.

Solution:

Step 1: Rewrite the equation in standard form

$$\begin{aligned}4x^2 - y^2 &= 16 \\ \frac{4x^2}{16} - \frac{y^2}{16} &= \frac{16}{16} \\ \frac{x^2}{4} - \frac{y^2}{16} &= 1 \\ \frac{x^2}{(2)^2} - \frac{y^2}{(4)^2} &= 1\end{aligned}$$

Step 2: Analysis.

Using the results of step 1, it is concluded that:

- a) since the x^2 term is positive, the transverse axis is horizontal
- b) since $a = \pm 2$, the vertices are $(-2, 0)$ and $(2, 0)$.
- c) since $b = \pm 4$, the endpoints of the conjugate axis are $(0, -4)$ and $(0, 4)$.
- d) since the transverse axis is horizontal, asymptote formula will be

$$y = \pm \frac{b}{a} x = \pm \frac{4}{2} x = \pm 2x .$$

Step 3: Plot and graph.

