

MATHEMATICAL INDUCTION

This section will discuss the principle of mathematical induction and then show its use through an example. Consider the following:

$$S_1 = 1 = 1^2$$

$$S_2 = 1 + 3 = 2^2$$

$$S_3 = 1 + 3 + 5 = 3^2$$

From this pattern, it is *assumed* that the n th term is $2n - 1$ and the sum of the first n terms is n^2 . It is important to remember that just because a rule or pattern seems to work for several values it cannot be decided that all values work without going through some sort of legitimate proof. The proof that may be used is the principle of mathematical induction, described below:

Let P_n be a statement involving the positive integer n . If

1. P_1 is true, and

*2. the truth of P_k implies the truth of P_{k+1} , for every positive integer k ,
then P_n must be true for all positive integers n .*

The following example will use this principle to prove the formula examined earlier.

Example 1: Use mathematical induction to prove the following formula:

$$S_n = 1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$$

Solution:

Step 1: Determine if the formula is true for $n = 1$.

$$S_1 = (2(1) - 1) = (1)^2$$

$$S_1 = (2 - 1) = 1$$

$$S_1 = 1 = 1$$

Since $S_1 = 1 = 1^2$ the formula is valid for $n = 1$.

The second part of the induction process has two steps. It must first be shown that the formula is valid for some integer k . Lastly the assumption is used to prove that the formula is valid for the next integer $k + 1$.

Example 1 (Continued):

Step 2: Determine S_k .

$$S_k = 1 + 3 + 5 + 7 + \dots + (2k - 1) = k^2$$

Step 3: Determine if $S_{k+1} = (k + 1)^2$ is true.

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + 7 + \dots + (2k - 1) + [2(k + 1) - 1] \\ &= [1 + 3 + 5 + 7 + \dots + (2k - 1)] + (2k + 2 - 1) \\ &= S_k + 2k + 1 \end{aligned}$$

[since step two determined that $S_k = k^2$]

$$\begin{aligned} S_{k+1} &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

By combining the above three steps, it may be concluded by mathematical induction that the formula is valid for all positive integer values of n .