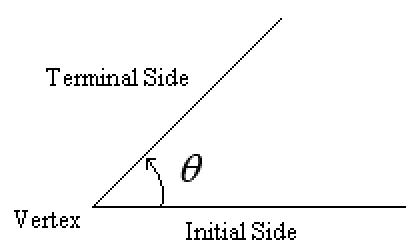
Angles and Radian Measure

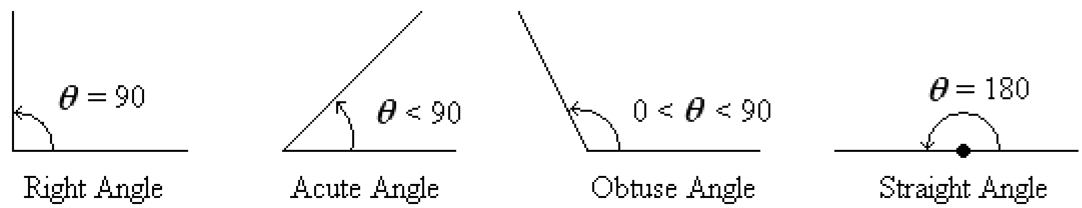
This section introduces the concept of angles and their measurement in relation to the study of trigonometry. In the previous two sections, we have dealt with trigonometric functions in terms of the unit circle with its center located at the origin of a rectangular coordinate system. We then moved around the circle in a counterclockwise or clockwise direction and measured the horizontal and vertical components of the point located somewhere on the perimeter of the circle. Various relationships between these components became the definitions of the trigonometric (or circular) functions.

When we study trigonometry in terms of angles, we need not think initially about the unit circle or a rectangular coordinate system. Instead, we are primarily concerned with the way angles are generated, and ultimately the angles and dimensions of right triangles in which two sides are perpendicular.

An angle is formed by starting with a finite straight line, called the initial side of the angle, and rotating another line around a point located at one end of the initial line, as shown below. After the rotation stops at some position, the second line is called the terminal side of the angle. The angle itself between the two sides is expressed as some symbol, such as the Greek letter θ (theta), and is valued in terms of some kind of rotational measure, such as degrees. The point at which the two sides meet is called the vertex of the angle. If the rotation occurs in a counterclockwise direction (as shown below), the angle is said to be positive, while if the rotation is clockwise, the angle is negative.



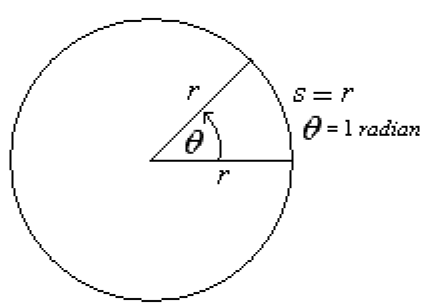
If the rotation ends at exactly a quarter rotation, producing initial and terminal sides which are perpendicular, the angle is said to be a right angle (θ = 90 degrees), as shown below. If the rotation is less than a quarter, the angle is said to be acute. If the rotation is greater than a quarter but less than a half rotation, the angle is obtuse. For exactly a half rotation, the angle is said to be a straight angle.



The most familiar unit for measuring angles is the degree. An angle containing exactly one complete rotation is defined to be equal to 360 degrees. A quarter rotation, producing a right angle, is equal to 360/4 or 90 degrees (90°). For measures which need to be stated more accurately than in whole degrees, the minute (’) and second (”) are frequently used. Similar to time measure, one degree equals 60 minutes, and one minute equals 60 seconds. As an example, an angle of 60 degrees, 35 minutes, and 52 seconds can be written using symbols as 60° 35’ 52”. Alternately, instead of subdividing degrees by minutes and seconds, decimal values of degrees are also frequently used.

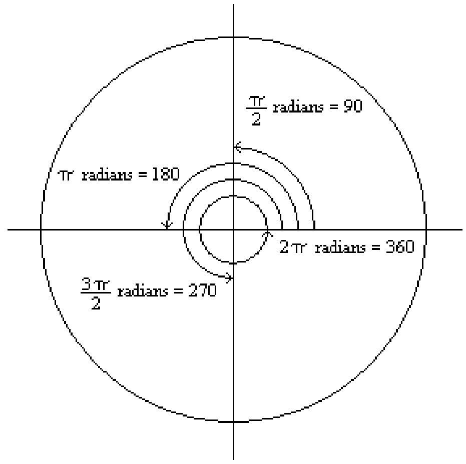
Angle measurement using degrees is common in everyday life as well as in surveying and navigation. However, another angular measure called the radian has frequent applications in mathematics, science, and engineering.

To define a radian, we start with a circle having a radius r, and then move around the circumference of the circle a distance s, such that s is equal to r, as shown below. The angle θ formed by the sides intersecting the circle at each end of the arc s is defined as 1 radian.



Since the circumference of a circle is equal to $2\pi r$, and an angle of 1 radian encloses an arc of the circle equal to r, the angle representing one complete rotation of the circle is equal to 2π radians. Thus, 2π radians in radian angle measure is equivalent to 360 degrees in degree angle measure. We can then easily compute the radian/degree equivalents for the three other quadrantal angles, as shown below:

2π radians = 360 degrees
π radians = 180 degrees
$\frac{\pi}{2}$ radians = 90 degrees
$\frac{3\pi}{2}$ radians = 270 degrees



If we wish to convert degrees to radians, or vice-versa, for any angle, we can use the conversion factor $\frac{2\pi}{360}$ for the calculation.

Example 1: Convert an angle of 150 degrees to radians.

Solution:

$$\begin{aligned}\text{The angle } \theta &= 150 \text{ degrees}\\ &= 150\left(\frac{2\pi}{360}\right) \text{ radians}\\ &= \frac{300\pi}{360}\\ &= \frac{5\pi}{6} \text{ radians}\end{aligned}$$

Example 2: Convert an angle of $\frac{4\pi}{3}$ radians to degrees.

Solution:

$$\begin{aligned}\text{The angle } \theta &= \frac{4\pi}{3} \text{ radians}\\ &= \frac{4\pi}{3}\left(\frac{360}{2\pi}\right) \text{ degrees}\\ &= \frac{1440\pi}{6\pi}\\ &= 240 \text{ degrees}\end{aligned}$$