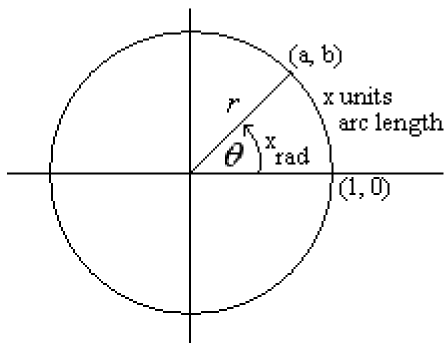
Trigonometric Functions of Any Angle

This section develops the trigonometric functions based on angle measurement and relates it to the concept of the circular or wrapping function developed previously, which relied on movement around the perimeter of a unit circle.

On the unit circle with radius $r = 1$, the distance around the perimeter or circumference of the circle is equal to 2π. Using the concept of angular measurement, it was determined that a complete rotation of a straight line around a point (the vertex of an angle) is equal to 360 degrees or 2π radians. Therefore, an angle measured in radians has the same value as the length of an arc on the unit circle created by this angle. If θ is an angle measured in radians and x is the length of the corresponding arc on the unit circle, then $\theta = x$, as shown below.



The six circular functions were defined based on the horizontal and vertical components of a point moving around the perimeter of a unit circle in a rectangular coordinate system. Now, we would like to define the same six functions as trigonometric functions in terms of angular measurement using radians. Since the angle θ (in radians) equals x (length of the arc on the unit circle), as shown above,

Trigonometric Function		Circular Function
$\sin\theta$	=	$\sin x$
$\cos\theta$	=	$\cos x$
$\tan\theta$	=	$\tan x$
$\sec\theta$	=	$\sec x$
$\csc\theta$	=	$\csc x$
$\cot\theta$	=	$\cot x$

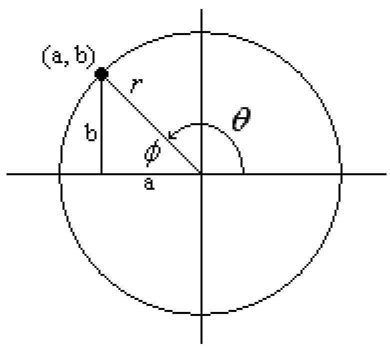
The trigonometric functions can be defined in terms of the horizontal and vertical components of the circular point shown above. If the horizontal and vertical components have values of a and b, respectively, and r is the radius of the circle, the trigonometric functions are defined as follows:

$$\sin\theta = \frac{b}{r} \qquad \cos\theta = \frac{a}{r}$$

$$\tan\theta = \frac{b}{a},\ a \neq 0 \qquad \cot\theta = \frac{a}{b},\ b \neq 0$$

$$\sec\theta = \frac{r}{a},\ a \neq 0 \qquad \csc\theta = \frac{r}{b},\ b \neq 0$$

It is important to realize that with these definitions of the trigonometric functions, we need not deal only with the unit circle having a radius $r = 1$. Instead, r can be the radius of any circle centered in a rectangular coordinate system, with a point on the circle having horizontal and vertical components a and b, respectively. The angle θ, said to be in **standard position**, is measured from the positive side of the horizontal axis to the terminal side of the angle, which may lie in any quadrant of the circle, as shown below. Components a and b can be positive or negative, depending on the quadrant in which they lie, while the radius r is always positive.

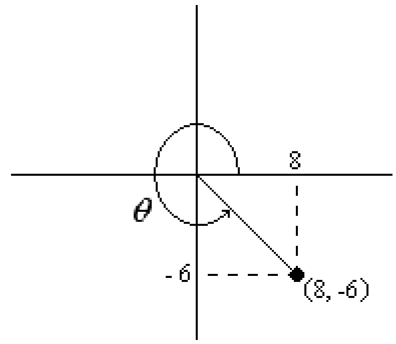


The triangle formed by sides a, b, and the radius is called a **reference triangle**. Since sides a and b are perpendicular, the triangle is a right triangle, and r is the hypotenuse. From the Pythagorean Theorem:

$$r = \sqrt{a^2 + b^2}$$

The acute angle represented here by the Greek letter ϕ (phi), formed between the horizontal axis and the terminal side of angle θ is called the **reference angle**.

Example 1: In the figure below, find the value of all six trigonometric functions of the angle θ.



Solution:

We see that $a = 8$ and $b = -6$. Therefore,

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{8^2 + (-6)^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

$$\sin\theta = \frac{b}{r} = \frac{-6}{10} = -\frac{3}{5} \qquad \cos\theta = \frac{a}{r} = \frac{8}{10} = \frac{4}{5}$$

$$\tan\theta = \frac{b}{a} = \frac{-6}{8} = -\frac{3}{4} \qquad \cot\theta = \frac{a}{b} = \frac{8}{-6} = -\frac{4}{3}$$

$$\sec\theta = \frac{r}{a} = \frac{10}{8} = \frac{5}{4} \qquad \csc\theta = \frac{r}{b} = \frac{10}{-6} = -\frac{5}{3}$$

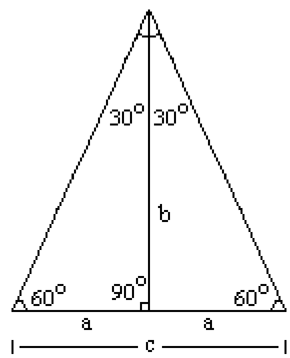
In this example, we were given the values of a and b and asked to evaluate each of the trigonometric functions. Alternatively, we could have been given the value of one of the trigonometric functions and the quadrant of the angle θ, and then been asked to find the value of the other five trigonometric functions. In this case, we can use the Pythagorean Theorem

$$r = \sqrt{a^2 + b^2} \text{ or } r^2 = a^2 + b^2$$

and substitution to relate r, a, and b to each other and solve for the other trigonometric functions.

We may remember from the study of geometry that the sum of all three interior angles of any triangle is equal to 180 degrees. If we are dealing with a right triangle, the angle opposite the hypotenuse is equal to 90 degrees. The sum of the other two angles must therefore equal 90 degrees. We can easily evaluate the relationships between the sides of two particular right triangles, one having acute angles of 30 and 60 degrees, and the other having acute angles of 45 and 45 degrees.

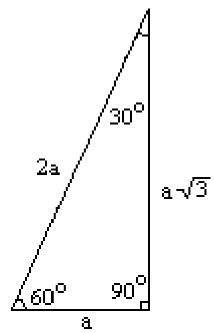
Consider the equilateral triangle shown below, in which all three sides are equal in length.



Each of the three interior angles is 60 degrees. If we draw a line from the top angle to the midpoint of the bottom side, this line is perpendicular to the bottom side and creates a 90 degree right angle. The original top angle is cut in half, creating two 30 degree angles. The side opposite each 30 degree angle is half the length of the original side of the equilateral triangle, so it is equal to half the length of the hypotenuse of the newly-created right triangle. If the hypotenuse is c, then the side a opposite the 30 degree angle is equal to $c/2$. This relationship can be rewritten as $c = 2a$. We can now use the Pythagorean Theorem to derive a relationship with the third side, the vertical line b.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ b^2 &= c^2 - a^2 \\ &= (2a)^2 - a^2 \\ &= 4a^2 - a^2 \\ &= 3a^2 \\ b &= \sqrt{3a^2} \\ &= a\sqrt{3} \end{aligned}$$

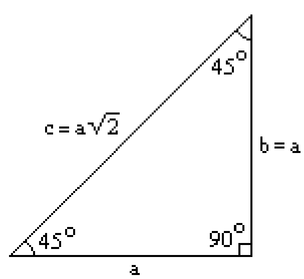
The sides of the 30-60 degree right triangle in terms of *a* are then as shown below.



For the 45-45 degree right triangle, we know that sides *a* and *b* are equal, since they are opposite from angles that are equal. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= a^2 + a^2 \\ &= 2a^2 \\ c &= \sqrt{2a^2} \\ &= a\sqrt{2} \end{aligned}$$

The sides of the 45-45 degree right triangle in terms of *a* are then as shown below.



Using the relationships between sides *a*, *b*, and *c* for the 30-60 and 45-45 degree right triangles derived above, we can easily determine the exact values of the trigonometric functions for these angles. In these definitions of the trigonometric functions, we let the hypotenuse *c* replace the radius *r* of the previously discussed circular functions.

$$\sin(30°) = \frac{a}{c} = \frac{a}{2a} = \frac{1}{2} \qquad \cos(30°) = \frac{b}{c} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\sin(60°) = \frac{b}{c} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2} \qquad \cos(60°) = \frac{a}{c} = \frac{a}{2a} = \frac{1}{2}$$

$$\sin(45°) = \frac{a}{c} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \cos(45°) = \frac{a}{c} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$