# Determinants and Cramer's Rule

This section will deal with how to find the determinant of a square matrix. Every square matrix can be associated with a real number known as its determinant. The determinant of a matrix, in this case a 2x2 matrix, is defined below:

$$\text{Given the matrix } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

The following example will show how to find the determinant of a 2x2 matrix and that these determinants may be positive, negative or zero.

**Example 1:** Find the determinants of the following matrices:

$$\textbf{a.)}\ \boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-3} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \qquad \textbf{b.)}\ \boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} \end{bmatrix} \qquad \textbf{c.)}\ \boldsymbol{C} = \begin{bmatrix} \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$$

$$a.)\ |A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$
$$= (2)(2) - (1)(-3)$$
$$= 4 - (-3)$$
$$= 4 + 3$$
$$= 7$$

$$b.)\ |B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$$
$$= (2)(2) - (4)(1)$$
$$= 4 - 4$$
$$= 0$$

$$c.)\ |C| = \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix}$$
$$= (0)(4) - (2)(3)$$
$$= 0 - 6$$
$$= -6$$

Solving a matrix greater than a 2x2 is simplified by using minors and cofactors of that matrix. Given a matrix, the minor of an element of that matrix is found using the remaining elements of the matrix after the row and column containing the original element are removed and then evaluated as in example 1. Example 2 will demonstrate this concept using a 3x3 matrix.

**Example 2:** Find all the minors of the following 3x3 matrix.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{4} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Step 1: Analysis.

The minors of the given matrix are labeled as:

$$\begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23} \\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33} \end{bmatrix}$$

in order to better identify their location with respect to their row and column.

Step 2: Set up and solve each minor.

The row and column containing each minor is eliminated and the remaining elements are arranged to solve for that elements minor.

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)(1) - (0)(2) = -1 - 0 = -1$$

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (3)(1) - (4)(2) = 3 - 8 = -5$$

$$M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = (3)(0) - (4)(-1) = 0 + 4 = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = (2)(1) - (0)(1) = 2 - 0 = 2$$

$$M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = (0)(1) - (4)(1) = 0 - 4 = -4$$

$$M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} = (0)(0) - (4)(2) = 0 - 8 = -8$$

**Example 2 (Continued):**

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = (2)(2) - (-1)(1) = 4 + 1 = 5$$

$$M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = (0)(2) - (3)(1) = 0 - 3 = -3$$

$$M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = (0)(-1) - (3)(2) = 0 - 6 = -6$$

Having found the minors of the 3x3 matrix, the cofactors are found by raising “-1” to a power equal to the sum of the elements row and column number and multiplying that value by the minor result (i.e. $M_{23}$ would result in -1 being raised to the 2+3 or $5^{th}$ power). Example 3 will demonstrate how to find the cofactors of the matrix given in example 2.

**Example 3:** Find the cofactors of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{4} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Step 1: Find the minors of the matrix.

Example 2 found the minors of the matrix to be:

$$\begin{array}{lll} \boldsymbol{M_{11} = -1} & \boldsymbol{M_{12} = -5} & \boldsymbol{M_{13} = 4} \\ \boldsymbol{M_{21} = 2} & \boldsymbol{M_{22} = -4} & \boldsymbol{M_{23} = -8} \\ \boldsymbol{M_{31} = 5} & \boldsymbol{M_{32} = -3} & \boldsymbol{M_{33} = -6} \end{array}$$

Step 2: Determine the cofactors using the formula $\boldsymbol{C_{i\,j} = (-1)^{i+j} M_{i\,j}}$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (-1) = (1)(-1) = -1$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (-5) = (-1)(-5) = 5$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 (4) = (1)(4) = 4$$

$$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 (2) = (-1)(2) = -2$$

$$C_{22} = (-1)^{2+2} M_{22} = (-1)^4 (-4) = (1)(-4) = -4$$

$$C_{23} = (-1)^{2+3} M_{23} = (-1)^5 (-8) = (-1)(-8) = 8$$

**Example 3 (Continued):**

$$C_{31} = (-1)^{3+1} M_{31} = (-1)^4 (5) = (1)(5) = 5$$
$$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-3) = (-1)(-3) = 3$$
$$C_{33} = (-1)^{3+3} M_{33} = (-1)^6 (-6) = (1)(-6) = -6$$

Having now found the cofactors, the determinant of the matrix may be found by finding the sum of the products of the entries of any row or column of the given matrix and their respective cofactor (i.e. $|A| = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$). Example 4 will show how to solve for the determinant of the matrix given in examples 2 and 3.

**Example 4:** Find the determinant of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{4} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Step 1: Find the cofactors of the given matrix.

From example 3 the cofactors of the matrix were found to be:

$$\begin{array}{lll} \boldsymbol{C_{11} = -1} & \boldsymbol{C_{12} = \ \ 5} & \boldsymbol{C_{13} = \ \ 4} \\ \boldsymbol{C_{21} = -2} & \boldsymbol{C_{22} = -4} & \boldsymbol{C_{23} = \ \ 8} \\ \boldsymbol{C_{31} = \ \ 5} & \boldsymbol{C_{32} = \ \ 3} & \boldsymbol{C_{33} = -6} \end{array}$$

Step 2: Select <u>*any*</u> row or column of the matrix and then, using the formula $|A| = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$, find the determinant. For this example, the second row and the third column will be evaluated.

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= (3)(-2) + (-1)(-4) + (2)(8) \\ &= -6 + 4 + 16 \\ &= 14 \end{aligned}$$

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \\ &= (1)(4) + (2)(8) + (1)(-6) \\ &= 4 + 16 - 6 \\ &= 14 \end{aligned}$$

Using the ideas demonstrated to this point, example 5 will demonstrate how to find the determinant of a 4x4 matrix.

**Example 5:** Find the determinant of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} & \mathbf{0} \\ \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{3} & \mathbf{4} & \mathbf{0} & \mathbf{2} \end{bmatrix}$.

Step 1: Analyze.

Recall that $|A| = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$ . For this example, the second column will be used even though the third column is the simplest to solve for. This means that for this example

$$\left|\boldsymbol{A}\right| = \mathbf{2}\boldsymbol{C}_{\mathbf{12}} + \mathbf{1}\boldsymbol{C}_{\mathbf{22}} + \mathbf{2}\boldsymbol{C}_{\mathbf{32}} + \mathbf{4}\boldsymbol{C}_{\mathbf{42}}$$

Step 2: Solve for $C_{12}$, $C_{22}$, $C_{32}$, and $C_{42}$.

$$\begin{aligned} C_{12} &= (-1)^3 \begin{vmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -\begin{vmatrix} -1 & 0 & 2 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -\left[ 2(-1)^4 \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} + 3(-1)^5 \begin{vmatrix} -1 & 0 \\ 3 & 0 \end{vmatrix} + 2(-1)^6 \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} \right] \\ &= -\left[ 2(0-0) - 3(0-0) + 2(0-0) \right] \\ &= 0 \end{aligned}$$

**Example 5 (Continued):**

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= 1(-1)^2 \begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix} + 0(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= 1(0-0) + 0(6-0) + 3(9-0)$$

$$= 0 + 0 + 27$$

$$= 27$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= -\left[ 1(-1)^2 \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} \right]$$

$$= -\left[ 1(0-0) + 1(6-0) + 3(6-0) \right]$$

$$= -(0 + 6 + 18)$$

$$= -24$$

$$C_{42} = (-1)^6 \begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 1(-1)^2 \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} + 0(-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= 1(0-0) + 1(9-0) + 0(6-0)$$

$$= 0 + 9 + 0$$

$$= 9$$

**Example 5 (Continued):**

Step 3: Solve for $|A|$.

$$
\begin{aligned}
|A| &= 2C_{12} + 1C_{22} + 2C_{32} + 4C_{42} \\
&= 2(0) + 1(27) + 2(-24) + 4(9) \\
&= 0 + 27 - 48 + 36 \\
&= 63 - 48 \\
&= 15
\end{aligned}
$$

This section will define and show how to use Cramer's rule to solve for systems of equations consisting of two equations and two variables or three equations and three variables.

Cramer's rule for a system of two equations with two variables is defined by:

$$\textit{Given} \quad \begin{matrix} a_{11}x + a_{12}y = k_1 \\ a_{21}x + a_{22}y = k_2 \end{matrix} \quad \textit{with} \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

$$\textit{then} \quad X = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \quad \textit{and} \quad Y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D}$$

The matrix of $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is known as the coefficient matrix.

**Example 6:** Given **4*x* – 2*y* = 10** and **3*x* – 5*y* = 11**, solve for x and y using Cramer's rule.

Solution:

Step 1: Analyze.

Using the given definition of Cramer's rule, the equations of **4*x* – 2*y* = 10** and **3*x* – 5*y* = 11** yield the elements:

$$a_{11} = 4\,;\, a_{12} = -2\,;\, k_1 = 10$$
$$\textit{and}$$
$$a_{21} = 3\,;\, a_{22} = -5\,;\, k_2 = 11$$

**Example 6 (Continued):**

Step 2: Substitute and solve for the coefficient matrix.

$$
\begin{aligned}
D &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\
&= \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} \\
&= (4)(-5)-(3)(-2) \\
&= -20-(-6) \\
&= -20+6 \\
&= -14
\end{aligned}
$$

Step 3: Solve for X and Y.

$$
\begin{aligned}
X &= \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \\
&= \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} \\
&= \frac{(10)(-5)-(11)(-2)}{-14} \\
&= \frac{-50+22}{-14} \\
&= \frac{-28}{-14} = 2
\end{aligned}
\qquad
\begin{aligned}
Y &= \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D} \\
&= \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} \\
&= \frac{(4)(11)-(3)(10)}{-14} \\
&= \frac{44-30}{-14} \\
&= \frac{14}{-14} = -1
\end{aligned}
$$

$$\therefore (x, y) = (2, -1)$$

Cramer's rule for solving a system of three equations and three unknowns is defined by:

$$\textit{Given}\quad \begin{matrix} a_{11}x + a_{12}y + a_{13}z = k_1 \\ a_{21}x + a_{22}y + a_{23}z = k_1 \\ a_{31}x + a_{32}y + a_{33}z = k_1 \end{matrix} \quad \textit{with}\quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

$$\textit{then}\quad X = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{D},\ Y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{D},\ Z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{D}$$

The matrix D is known as the coefficient matrix.

**Example 7:** Use Cramer's rule to solve for the following system:

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x + 0y + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

Solution:

Step 1: Analyze.

$$\begin{matrix} a_{11} = -1 & a_{12} = 2 & a_{13} = -3 & k_1 = 1 \\ a_{21} = 2 & a_{22} = 0 & a_{23} = 1 & k_2 = 0 \\ a_{31} = 3 & a_{32} = -4 & a_{33} = 4 & k_3 = 2 \end{matrix}$$

Step 2: Find the coefficient matrix, D.

$$\begin{aligned} D &= \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} \\ &= (-1)(-1)^2 \begin{vmatrix} 0 & 1 \\ -4 & 4 \end{vmatrix} + (2)(-1)^3 \begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} + (3)(-1)^4 \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} \\ &= (-1)(1)\left[(0)-(-4)\right] + (2)(-1)\left[(8)-(12)\right] + (3)(1)\left[(2)-(0)\right] \\ &= (-1)(4) + (-2)(-4) + (3)(2) \\ &= -4 + 8 + 6 \\ &= 10 \end{aligned}$$

**Example 7 (Continued):**

Step 3: Solve for x, y and z.

$$x=\frac{\begin{vmatrix}1 & 2 & -3\\ 0 & 0 & 1\\ 2 & -4 & 4\end{vmatrix}}{10}$$

$$=\frac{(-3)(-1)^4\begin{vmatrix}0 & 0\\ 2 & -4\end{vmatrix}+(1)(-1)^5\begin{vmatrix}1 & 2\\ 2 & -4\end{vmatrix}+(4)(-1)^6\begin{vmatrix}1 & 2\\ 0 & 0\end{vmatrix}}{10}$$

$$=\frac{(-3)(1)\left[(0)-(0)\right]+(1)(-1)\left[(-4)-(4)\right]+(4)(1)\left[(0)-(0)\right]}{10}$$

$$=\frac{(-3)(0)+(-1)(-8)+(4)(0)}{10}$$

$$=\frac{0+8+0}{10}$$

$$=\frac{8}{10}=\frac{4}{5}$$

$$y=\frac{\begin{vmatrix}-1 & 1 & -3\\ 2 & 0 & 1\\ 3 & 2 & 4\end{vmatrix}}{10}$$

$$=\frac{(-3)(-1)^4\begin{vmatrix}2 & 0\\ 3 & 2\end{vmatrix}+(1)(-1)^5\begin{vmatrix}-1 & 1\\ 3 & 2\end{vmatrix}+(4)(-1)^6\begin{vmatrix}-1 & 1\\ 2 & 0\end{vmatrix}}{10}$$

$$=\frac{(-3)(1)\left[(4)-(0)\right]+(1)(-1)\left[(-2)-(3)\right]+(4)(1)\left[(0)-(2)\right]}{10}$$

$$=\frac{(-3)(4)+(-1)(-5)+(4)(-2)}{10}$$

$$=\frac{-12+5-8}{10}$$

$$=-\frac{15}{10}=-\frac{3}{2}$$

**Example 7 (Continued):**

$$z = \frac{\begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix}}{10}$$

$$= \frac{(-1)(-1)^2 \begin{vmatrix} 0 & 0 \\ -4 & 2 \end{vmatrix} + (2)(-1)^3 \begin{vmatrix} 2 & 1 \\ -4 & 2 \end{vmatrix} + (3)(-1)^4 \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix}}{10}$$

$$= \frac{(-1)(1)\left[(0)-(0)\right] + (2)(-1)\left[(4)-(-4)\right] + (3)(1)\left[(0)-(0)\right]}{10}$$

$$= \frac{(-1)(0) + (-2)(8) + (3)(0)}{10}$$

$$= \frac{0-16+0}{10}$$

$$= -\frac{16}{10} = -\frac{8}{5}$$

Step 4: Analyze.

The solutions found were $x = \frac{4}{5}$, $y = -\frac{3}{2}$ *and* $z = -\frac{8}{5}$.

This indicates that the solution set or point of interception for the three given lines is:

$$(x, y, z) = \left(\frac{4}{5}, -\frac{3}{2}, -\frac{8}{5}\right)$$