Multiplicative Inverses of Matrices and Matrix Equations

This section will deal with how to find the Identity of a matrix and how to find the inverse of a square matrix.

A square matrix is one in which the number of rows and columns of the matrix are equal in number. Matrices of this nature are the only ones that have an identity. The identity matrix is one in which the principle diagonal consists of 1's and the remaining values of the matrix are zeros. The examples below illustrate this.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is important to understand that when a square matrix (M) is multiplied by identity matrix (I) the solution is the original matrix: $MI = IM = M$. Another property is that when a matrix (M) is multiplied cy its inverse (M^{-1}) the product is the identity (I).

$$M * M^{-1} = I$$

The following examples will show a method to solve for the inverse of a matrix.

Example 1: Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$.

Step 1: Analysis.

Recall that $M * M^{-1} = I$. Therefore:

$$A * B \text{ (} \textit{the inverse of } A, A^{-1} \text{) } = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (} \textit{the identitiy} \text{)}$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x & a \\ y & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 1 (Continued):

Step 2: Find the linear equations.

Multiplying the entries from step 1 yields:

$$\begin{bmatrix} x+4y & a+4b \\ -x-3y & -a-3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries yields the following systems:

$$\begin{aligned} x+4y&=1 \\ -x-3y&=0 \end{aligned} \qquad \begin{aligned} a+4b&=0 \\ -a-3b&=1 \end{aligned}$$

Step 3: Solve for *a*, *b*, *x*, and *y*.

Using the first system, *x* and *y* are solved for.

$$\begin{array}{r} x+4y=1 \\ \underline{-x-3y=0} \\ y=1 \end{array} \qquad \begin{aligned} x+4y&=1 \\ x+4(1)&=1 \\ x+4&=1 \\ x&=-3 \end{aligned}$$

Using the second system, *a* and *b* are solved for.

$$\begin{array}{r} a+4b=0 \\ \underline{-a-3b=1} \\ b=1 \end{array} \qquad \begin{aligned} a+4b&=0 \\ a+4(1)&=0 \\ a+4&=0 \\ a&=-4 \end{aligned}$$

Step 4: Solution.

From step 3 $A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

This section will deal with how to use an identity or inverse matrix to solve a system of equations. (Recall that if the system contains fewer or more variables than equations, then the Gauss-Jordan method of elimination is used.)

It is understood that a system of linear equations can have exactly one solution, an infinite number of solutions, or no solution. In a square system, the following can be used to determine whether the system has a unique solution:

Systems of Equations with Unique Solutions:
If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

The advantage of using the inverse method as opposed to Gaussian elimination is apparent when several systems with the same coefficient matrix are being solved for. The following example will demonstrate this.

Example 2: Use an inverse matrix to solve the following systems:

a.) $\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned}$

b.) $\begin{aligned} 2x + 3y + z &= 4 \\ 3x + 3y + z &= 8 \\ 2x + 4y + z &= 5 \end{aligned}$

c.) $\begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$

Solution:

Step 1: Analyze.

Note that the coefficient matrix for each system is $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Example 2 (Continued):

Step 2: Using Gauss-Jordan elimination find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] R_2 + \left(-R_1\right) = R_1$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] R_2 + \left(-3R_1\right) = R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & 3 & -2 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] R_3 + \left(-2R_1\right) = R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & 3 & -2 & 0 \\ 0 & 4 & 1 & 2 & -2 & 1 \end{array}\right] R_3 + \left(-R_2\right) = R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 4 & 1 & 2 & -2 & 1 \end{array}\right] R_3 + \left(-4R_2\right) = R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Example 2 (Continued):

Step 3: Solve each system by matrix multiplication.

a.) $X = A^{-1}B =$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} =$$

$$(-1)(-1)+(1)(1)+(0)(-2)=1+1+0=2=x$$
$$(-1)(-1)+(0)(1)+(1)(-2)=1+0+(-2)=-1=y$$
$$(6)(-1)+(-2)(1)+(-3)(-2)=-6+(-2)+6=-2=z$$
$$\therefore x=2,\ y=-1,\ z=-2$$

b.) $X = A^{-1}B =$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} =$$

$$(-1)(4)+(1)(8)+(0)(5)=-4+8+0=4=x$$
$$(-1)(4)+(0)(8)+(1)(5)=-4+0+(5)=1=y$$
$$(6)(4)+(-2)(8)+(-3)(5)=24+(-16)+(-15)=-7=z$$
$$\therefore x=4,\ y=1,\ z=-7$$

c.) $X = A^{-1}B$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} =$$

$$(-1)(0)+(1)(0)+(0)(0)=0+0+0=0=x$$
$$(-1)(0)+(0)(0)+(1)(-2)=0+0+0=0=y$$
$$(6)(0)+(-2)(0)+(-3)(0)=0+0+0=0=z$$
$$\therefore x=0,\ y=0,\ z=0$$