

Dividing Polynomials; Remainder and Factor Theorems

In this section we will learn how to divide polynomials, an important tool needed in factoring them. This will begin our *algebraic* study of polynomials.

Dividing by a Monomial:

Recall from the previous section that a monomial is a single term, such as $6x^3$ or -7 . To divide a polynomial by a monomial, divide each term in the polynomial by the monomial, and then write each quotient in lowest terms.

Example 1: Divide $9x^4 + 3x^2 - 5x + 6$ by $3x$.

Solution:

Step 1: Divide each term in the polynomial $9x^4 + 3x^2 - 5x + 6$ by the monomial $3x$.

$$\frac{9x^4 + 3x^2 - 5x + 6}{3x} = \frac{9x^4}{3x} + \frac{3x^2}{3x} - \frac{5x}{3x} + \frac{6}{3x}$$

Step 2: Write the result in lowest terms.

$$\frac{9x^4}{3x} + \frac{3x^2}{3x} - \frac{5x}{3x} + \frac{6}{3x} = 3x^3 + x - \frac{5}{3} + \frac{2}{x}$$

Thus, $9x^4 + 3x^2 - 5x + 6$ divided by $3x$ is equal to $3x^3 + x - \frac{5}{3} + \frac{2}{x}$

Long Division of Polynomials:

To divide a polynomial by a polynomial that is not a monomial we must use long division. Long division for polynomials is very much like long division for numbers. For example, to divide $3x^2 - 17x - 25$ (the **dividend**) by $x - 7$ (the **divisor**), we arrange our work as follows.

$$\begin{array}{r}
 \boxed{\text{quotient}} \\
 \downarrow \\
 3x+4 \\
 \boxed{\text{divisor}} \rightarrow x-7 \overline{) 3x^2 - 17x - 25} \leftarrow \boxed{\text{dividend}} \\
 \underline{3x^2 - 21x} \qquad \text{Multiply divisor by } 3x \\
 4x - 25 \qquad \text{Subtract, then "bring down" } -25 \\
 \underline{4x - 28} \qquad \text{Multiply divisor by } 4 \\
 3 \qquad \leftarrow \boxed{\text{remainder}} \text{ Subtract}
 \end{array}$$

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the **remainder**, and the top line contains the **quotient**. The result of the division can be interpreted in either of two ways

$$\frac{3x^2 - 17x - 25}{x - 7} = 3x + 4 + \frac{3}{x - 7}$$

or

$$3x^2 - 17x - 25 = (x - 7)(3x + 4) + 3$$

We summarize what happens in any long division problem in the following theorem.

Division Algorithm:

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

where $R(x)$ is either 0 or of less degree than the degree of $D(x)$. The polynomials $P(x)$ and $D(x)$ are called the **dividend** and the **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

Example 2: Let $P(x) = 3x^2 + 17x + 10$ and $D(x) = 3x + 2$. Using long division, find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

Solution:

Step 1: Write the problem, making sure that both polynomials are written in descending powers of the variables.

$$3x + 2 \overline{) 3x^2 + 17x + 10}$$

Example 2 (Continued):

Step 2: Divide the first term of $P(x)$ by the first term of $D(x)$.

Since $\frac{3x^2}{3x} = x$, place this result above the division line.

$$\begin{array}{r} x \quad \leftarrow \text{Result of } \frac{3x^2}{3x} \\ 3x+2 \overline{)3x^2+17x+10} \end{array}$$

Step 3: Multiply $3x + 2$ and x , and write the result below $3x^2 + 17x + 10$.

$$\begin{array}{r} x \\ 3x+2 \overline{)3x^2+17x+10} \\ \underline{3x^2+2x} \quad \leftarrow x(3x+2) = 3x^2+2x \end{array}$$

Step 4: Now subtract $3x^2 + 2x$ from $3x^2 + 17x$. Do this by mentally changing the signs on $3x^2 + 2x$ and adding.

$$\begin{array}{r} x \\ 3x+2 \overline{)3x^2+17x+10} \\ \underline{3x^2+2x} \\ 15x \quad \leftarrow \text{Subtract} \end{array}$$

Step 5: Bring down 10 and continue by dividing $15x$ by $3x$.

$$\begin{array}{r} x+5 \quad \leftarrow \frac{15x}{3x} = 5 \\ 3x+2 \overline{)3x^2+17x+10} \\ \underline{3x^2+2x} \\ 15x+10 \quad \leftarrow \text{Bring down 10} \\ \underline{15x+10} \quad \leftarrow 5(3x+2) = 15x+10 \\ 0 \quad \leftarrow \text{Subtract} \end{array}$$

Example 2 (Continued):

Step 6: The process is complete at this point because we have a zero in the final row. From the long division table we see that $Q(x) = x + 5$ and $R(x) = 0$, so

$$3x^2 + 17x + 10 = (3x + 2)(x + 5) + 0$$

Note that since there is no remainder, this quotient could have been found by factoring and writing in lowest terms.

Example 3: Find the quotient and remainder of $\frac{4x^3 - 3x - 2}{x + 1}$ using long division.

Solution:

Step 1: Write the problem, making sure that both polynomials are written in descending powers of the variables. Add a term with 0 coefficient as a place holder for the missing x^2 term.

$$\begin{array}{r} \overline{)4x^3 + 0x^2 - 3x - 2} \\ \end{array}$$

Missing term
↓

Step 2: Start with $\frac{4x^3}{x} = 4x^2$.

$$\begin{array}{r} \overline{)4x^3 + 0x^2 - 3x - 2} \\ \underline{4x^3} \\ \end{array}$$

$\leftarrow \frac{4x^3}{x} = 4x^2$

$$\begin{array}{r} \overline{)4x^3 + 0x^2 - 3x - 2} \\ \underline{4x^3 + 4x^2} \\ \end{array}$$

$\leftarrow 4x^2(x+1)$

Step 3: Subtract by changing the signs on $4x^3 + 4x^2$ and adding. Then Bring down the next term.

$$\begin{array}{r} \overline{)4x^3 + 0x^2 - 3x - 2} \\ \underline{4x^3 + 4x^2} \\ - 3x - 2 \end{array}$$

\leftarrow Subtract and bring down $-3x$

Example 3 (Continued):

Step 4: Now continue with $\frac{-4x^2}{x} = -4x$.

$$\begin{array}{r} 4x^2 - 4x \quad \leftarrow \frac{-4x^2}{x} = -4x \\ x+1 \overline{) 4x^3 + 0x^2 - 3x - 2} \\ \underline{4x^3 + 4x^2} \\ -4x^2 - 3x \\ \underline{-4x^2 - 4x} \\ x - 2 \quad \leftarrow \text{Subtract and bring down } -2 \end{array}$$

Step 5: Finally, $\frac{x}{x} = 1$.

$$\begin{array}{r} 4x^2 - 4x + 1 \quad \leftarrow \frac{x}{x} = 1 \\ x+1 \overline{) 4x^3 + 0x^2 - 3x - 2} \\ \underline{4x^3 + 4x^2} \\ -4x^2 - 3x \\ \underline{-4x^2 - 4x} \\ x - 2 \\ \underline{x + 1} \quad \leftarrow 1(x+1) \\ -3 \quad \leftarrow \text{Subtract} \end{array}$$

Step 6: The process is complete at this point because -3 is of lesser degree than the divisor $x + 1$. Thus, the quotient is $4x^2 - 4x + 1$ and the remainder is -3 , and

$$\frac{4x^3 - 3x - 2}{x + 1} = 4x^2 - 4x + 1 + \frac{-3}{x + 1}.$$

Synthetic Division:

Synthetic division is a shortcut method of performing long division that can be used when the divisor is a first degree polynomial of the form $x - c$. In synthetic division we write only the essential part of the long division table. To illustrate, compare these long division and synthetic division tables, in which we divide $3x^3 - 4x + 2$ by $x - 1$:

Long Division

$$\begin{array}{r}
 \overline{) 3x^3 + 0x^2 - 4x + 2} \\
 \underline{3x^3 - 3x^2} \\
 3x^2 - 4x \\
 \underline{3x^2 - 3x} \\
 -x + 2 \\
 \underline{-x + 1} \\
 1
 \end{array}$$

← remainder

Synthetic Division

$$\begin{array}{r|rrrr}
 1 & 3 & 0 & -4 & 2 \\
 & & 3 & 3 & -1 \\
 \hline
 & 3 & 3 & -1 & 1
 \end{array}$$

↑ quotient ↑ remainder

Note that in synthetic division we abbreviate $3x^3 - 4x + 2$ by writing only the coefficients: 3 0 -4 2, and instead of $x - 1$, we simply write 1. (Writing 1 instead of -1 allows us to add instead of subtract, but this changes the sign of all the numbers that appear in the yellow boxes.)

To divide $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ by $x - c$, we proceed as follows:

$$\begin{array}{r|rrrrrrrr}
 c & a_n & a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_2 & a_1 & a_0 \\
 & & \nearrow cb_{n-1} & \nearrow cb_{n-2} & \nearrow cb_{n-3} & \dots & \nearrow cb_2 & \nearrow cb_1 & \nearrow cb_0 \\
 \hline
 & b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \dots & b_1 & b_0 & r
 \end{array}$$

Here $b_{n-1} = a_n$, and each number in the bottom row is obtained by adding the numbers above it. The remainder is r and the quotient is

$$b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_1 x + b_0$$

Example 4: Find the quotient and the remainder of $\frac{x^4 - 7x^2 - 6x}{x + 2}$ using synthetic division.

Solution:

Step 1: We put $x + 2$ in the form $x - c$ by writing it as $x - (-2)$. Use this and the coefficients of the polynomial to obtain

$$-2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right.$$

Note that we used 0 as the coefficient of any missing powers.

Step 2: Next, bring down the 1.

$$\begin{array}{r} -2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right. \\ \hline 1 \end{array}$$

Step 3: Now, multiply -2 by 1 to get -2 , and add it to the 0 in the first row. The result is -2 .

$$\begin{array}{r} -2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right. \\ \hline -2 \\ \hline 1 \end{array}$$

Step 4: Next, $-2(-2) = 4$. Add this to the -7 in the first row.

$$\begin{array}{r} -2 \left| \begin{array}{ccccc} 1 & 0 & -7 & -6 & 0 \end{array} \right. \\ \hline -2 \\ \hline 1 -3 \end{array}$$

Example 4 (Continued):

Step 5: $-2(-3) = 6$. Add this to the -6 in the first row.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & & & -2 & 4 & 6 \\ \hline & 1 & -2 & -3 & 0 & & \end{array}$$

Step 6: Finally, $-2(0) = 0$, which is added to 0 to get 0.

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -7 & -6 & 0 \\ & & & & -2 & 4 & 6 & 0 \\ \hline & 1 & -2 & -3 & 0 & 0 & & \end{array}$$

The coefficients of the quotient polynomial and the remainder are read directly from the bottom row. Also, the degree of the quotient will always be one less than the degree of the dividend. Thus, $Q(x) = x^3 - 2x^2 - 3x$ and $R(x) = 0$.

The Remainder and Factor Theorems:

Synthetic division can be used to find the values of polynomials in a sometimes easier way than substitution. This is shown by the next theorem.

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Example 5: Use synthetic division and the Remainder Theorem to evaluate $P(c)$ if $P(x) = x^3 - 4x^2 + 2x - 1$, $c = -1$.

Solution:

Step1: First we will use synthetic division to divide $P(x) = x^3 - 4x^2 + 2x - 1$ by $x - (-1)$.

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 2 & -1 \\ & & -1 & 5 & -7 \\ \hline & 1 & -5 & 7 & -8 \end{array}$$

Step 2: Since the remainder when $P(x)$ is divided by $x - (-1) = x + 1$ is -8 , by the Remainder Theorem, $P(-1) = -8$.

We learned that if c is a *zero* of P , then $x - c$ is a *factor* of $P(x)$. The next theorem restates this fact in a more useful way.

Factor Theorem: c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

Example 6: Use the Factor Theorem to show that $x + \frac{1}{2}$ is a factor of $P(x) = 2x^3 + 5x^2 + 4x + 1$.

Solution:

In order to show that $x + \frac{1}{2}$ is a factor of $P(x) = 2x^3 + 5x^2 + 4x + 1$, we must show that $-\frac{1}{2}$ is a zero of P , or that $P\left(-\frac{1}{2}\right) = 0$. We will use synthetic division and the Remainder Theorem to do this.

Step 1: Use synthetic division to divide $P(x) = 2x^3 + 5x^2 + 4x + 1$ by $x - \left(-\frac{1}{2}\right)$.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 5 & 4 & 1 \\ & & -1 & -2 & -1 \\ \hline & 2 & 4 & 2 & 0 \end{array}$$

Step 2: Since the remainder is 0, by the Remainder Theorem, we know $P\left(-\frac{1}{2}\right) = 0$.

Step 3: Finally, since $P\left(-\frac{1}{2}\right) = 0$, we know that $-\frac{1}{2}$ is a zero of P , by definition. Hence, by the Factor Theorem, $x + \frac{1}{2}$ is a factor of $P(x) = 2x^3 + 5x^2 + 4x + 1$.

Example 7: Find a polynomial of degree 3 that has zeros -2 , 0 , and 4 , and in which the coefficient of x is -4 .

Solution:

Step 1: We will begin finding our polynomial by looking at the zeros we are given. In order for -2 to be a zero of P , $x - (-2)$ must be a factor of $P(x)$, by the Factor Theorem. By the same argument $x - 0$ and $x - 4$ are also factors of $P(x)$.

Step 2: Now we will build a polynomial out of the factors we have found. A polynomial of degree 3 with factors -2 , 0 , and 4 could be

$$P(x) = (x - (-2))(x - 0)(x - 4) = x(x + 2)(x - 4).$$

Step 3: By expanding the polynomial, we can inspect the coefficients.

$$\begin{aligned} P(x) &= x(x + 2)(x - 4) \\ &= (x^2 + 2x)(x - 4) \\ &= x^3 - 2x^2 - 8x \end{aligned}$$

Step 4: We want a polynomial with -4 as the coefficient of x . The polynomial in the previous step has -8 as the coefficient of x . So, if we multiply the previous polynomial by $\frac{1}{2}$, we will obtain our desired answer.

$$\begin{aligned} P(x) &= \frac{1}{2}(x^3 - 2x^2 - 8x) \\ &= \frac{1}{2}x^3 - x^2 - 4x \end{aligned}$$

Note that the $P(x)$ in the current step is not the same $P(x)$ as in the previous step.

Step 5: Thus, a polynomial of degree 3 that has zeros -2 , 0 , and 4 , and with the coefficient of x as -4 , is

$$P(x) = \frac{1}{2}x^3 - x^2 - 4x = \frac{1}{2}x(x + 2)(x - 4).$$