Geometric Sequences

A geometric sequence (or progression) may be defined as:

> ***A sequence $\{a_n\}$ where each pair of consecutive terms has the same nonzero ratio, $r = \frac{a_i}{a_{i-1}}$, $r \neq 0$. The number r is called the common ratio of the sequence.***

Note that the definition will give the recursive formula $a_{i+1} = ra_i$. The following example will show how to find the common ratio of a geometric sequence.

Example 1: Find the common ratio of the geometric sequence $a_n = \frac{2}{3^n}$.

Solution:

Step 1: Substitute into the ratio formula.

The sequence is substituted into the definition.

$$r = \frac{a_i}{a_{i-1}} = \frac{2/3^i}{2/3^{i-1}}$$

Step 2: Solve for r.

$$r = \frac{2/3^i}{2/3^{i-1}}$$

$$r = \left(\frac{2}{3^i}\right)\left(\frac{3^{i-1}}{2}\right) = \frac{3^{i-1}}{3^i} = \frac{1}{3^{i-(i-1)}}$$

$$r = 1/3$$

The following is a definition for nth term of a geometric sequence.

> The nth term of a geometric sequence, whose first term is a_1 and whose common ratio is r, is given by the formula $a_n = a_1 r^{n-1}$.

The three examples following will show various means to find the nth term of geometric sequences.

Example 2: Find the first five terms of the geometric sequence whose first term is a_1 = 3 and whose common ratio is r = –2.

Solution:

Step 1: Substitute the given information into the definition and solve.

$$\begin{aligned}
a_1 &= 3(Given) \\
a_2 &= 3(-2)^{2-1=1} = 3(-2) = -6 \\
a_3 &= 3(-2)^{3-1=2} = 3(4) = 12 \\
a_4 &= 3(-2)^{4-1=3} = 3(-8) = -24 \\
a_5 &= 3(-2)^{5-1=4} = 3(16) = 48
\end{aligned}$$

Example 3: Find the twelfth term of the geometric sequence 5, -15 , 45 , … .

Solution:

Step 1: Analysis.

The terms given are a_1 = 5 and n = 12. The common ratio, using the definition given is $r = \frac{-15}{5} = -3$.

Step 2: Substitute and solve.

$$\begin{aligned}
a_n &= a_1 r^{n-1} \\
a_{12} &= 5(-3)^{12-1=11} \\
a_{12} &= 5(-177,147) \\
a_{12} &= -885,735
\end{aligned}$$

Example 4: The fourth term of a geometric sequence is 125, and the tenth term is $\frac{125}{64}$. Find the fourteenth term.

Solution:

Step 1: Analysis.

Using the nth term formula for geometric sequences the given values may be rewritten as:

$$a_4 = a_1 r^3 = 125 \qquad \textit{and} \qquad a_{10} = a_1 r^9 = \frac{125}{64}$$

Step 2: Solve for a_1.

Using the first equation from step 1, a_1 is solved for.

$$a_1 r^3 = 125$$
$$a_1 = \frac{125}{r^3}$$

Step 3: Substitute.

The value of a_1 found in step 2 is substituted into the second equation found in step 1 to solve for r.

$$a_1 r^9 = \frac{125}{64}$$
$$\left(\frac{125}{r^3}\right) r^9 = \frac{125}{64}$$
$$125 r^6 = \frac{125}{64}$$
$$r^6 = \frac{1}{64}$$
$$\sqrt[6]{r^6} = \sqrt[6]{\frac{1}{64}}$$
$$r = \frac{1}{2}$$

Example 4 (Continued):

Step 4: Substitution.

The value of r found in step 3 is substituted back into the first equation to solve for a_1

$$a_1 r^3 = 125$$
$$a_1 \left(\frac{1}{2}\right)^3 = 125$$
$$a_1 = (125)\left(2^3\right) = (125)(8)$$
$$a_1 = 1000$$

Step 5: Substitute and solve for a_{14}.

$$a_n = a_1 r^{n-1}$$
$$a_{14} = (1000)\left(\frac{1}{2}\right)^{14-1=13}$$
$$a_{14} = \frac{1000}{8192} = \frac{125}{1024}$$

To find the sum of a geometric series, either of the following two formulas may be used:

$$S_n = \frac{a_1 - a_1 r^n}{1-r} \quad or\ S_n = \frac{a_1 - r a_n}{1-r} \ \ ; r \neq 1$$

Example 5 will show how to find the sum of a geometric series.

Example 5: Find the sum of the first seven terms of the geometric series 2 + 6 + 18 + … .

Solution:

Step 1: Analysis.

The terms $a_1 = 2$ and $n = 7$ are given. The common ratio is found to be $r = \frac{6}{2} = 3$.

Example 5 (Continued):

Step 2: Substitute and solve.

$$S_n = \frac{a_1 - a_1 r^n}{1-r}$$
$$S_7 = \frac{2-\left[(2)\left(3^7\right)\right]}{1-3}$$
$$= \frac{2-\left[(2)(2187)\right]}{-2}$$
$$= \frac{2-4374}{-2} = \frac{-4372}{-2}$$
$$= 2186$$

The final topic to be covered is that concerning the sum of an infinite series. The definition of the sum of an infinite geometric series is:

$$\textit{If } |r| < 1, \textit{ then the infinite geometric series has the sum } S = \frac{a_1}{1-r}.$$

The final example shows its use.

Example 6: Find the sum of the infinite geometric series whose first term is 4 and whose common ratio is -0.6.

Solution:

Step 1: Substitute and solve.

$$S = \frac{a_1}{1-r}$$
$$S = \frac{4}{1-(-0.6)}$$
$$= \frac{4}{1+0.6}$$
$$= \frac{4}{1.6}$$
$$= 2.5$$