Systems of Linear Equations: Determinants

This section will deal with how to find the determinant of a square matrix. Every square matrix can be associated with a real number known as its determinant. The determinant of a matrix, in this case a 2x2 matrix, is defined below:

$$\text{Given the matrix } \mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$\det(\mathrm{A}) = |\mathrm{A}| = a_{11}a_{22} - a_{21}a_{12}$$

The following example will show how to find the determinant of a 2x2 matrix and that these determinants may be positive, negative or zero.

Example 1: Find the determinants of the following matrices:

$$\boldsymbol{a.)}\ \ \boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-3} \\ \mathbf{1} & \mathbf{2} \end{bmatrix} \qquad \boldsymbol{b.)}\ \ \boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} \end{bmatrix} \qquad \boldsymbol{c.)}\ \ \boldsymbol{C} = \begin{bmatrix} \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$$

$$\begin{aligned} a.)\ \ |A| &= \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} \\ &= (2)(2) - (1)(-3) \\ &= 4 - (-3) \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

$$\begin{aligned} b.)\ \ |B| &= \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \\ &= (2)(2) - (4)(1) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} c.)\ \ |C| &= \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} \\ &= (0)(4) - (2)(3) \\ &= 0 - 6 \\ &= -6 \end{aligned}$$

Solving a matrix greater than a 2x2 is simplified by expanding it into a series of 2x2 determinants. The expansion process is accomplished by first selecting row 1 or column 1. Each element in the selected row or column is multiplied by a 2x2 determinant that is found using the remaining elements of the matrix after the row and column containing the original element are removed.

When writing out the expansion of the 3x3 determinant you must remember to always alternate the signs between each term.

Expansion using row 1 of a 3x3 determinant

$$\begin{vmatrix} a_1 & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} \cancel{a_1} & b_1 & \cancel{c_1} \\ a_2 & \cancel{b_2} & c_2 \\ a_3 & \cancel{b_3} & c_3 \end{vmatrix} \qquad \begin{vmatrix} \cancel{a_1} & \cancel{b_1} & c_1 \\ a_2 & b_2 & \cancel{c_2} \\ a_3 & b_3 & \cancel{c_3} \end{vmatrix}$$

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \qquad b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \qquad c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Example 2: Find the determinant by expanding it using row 1.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{4} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Example 2 (Continued):

Solution:

Step 1: Expand the 3x3 determinant into a sum of 2x2 determinants.

$$\begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix}$$

Step 2: Evaluate the 2x2 determinants.

$$\begin{aligned} \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} &= 0\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} \\ &= 0\left[(-1)(1)-(0)(2)\right]-2\left[(3)(1)-(4)(2)\right]+1\left[(3)(0)-(4)(-1)\right] \\ &= 0(-1-0)-2(3-8)+1(0+4) \\ &= 0(-1)-2(-5)+1(4) \\ &= 0+10+4 \\ &= 14 \end{aligned}$$

This next section will define and show how to use Cramer's rule to solve for systems of equations consisting of two equations and two variables or three equations and three variables.

Cramer's rule for a system of two equations with two variables is defined by:

$$\boldsymbol{Given} \quad \begin{matrix} \boldsymbol{a_{11}x + a_{12}y = k_1} \\ \boldsymbol{a_{21}x + a_{22}y = k_2} \end{matrix} \quad \boldsymbol{with} \quad \boldsymbol{D} = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{vmatrix} \neq \boldsymbol{0}$$

$$\boldsymbol{then} \quad \boldsymbol{X} = \frac{\begin{vmatrix} \boldsymbol{k_1} & \boldsymbol{a_{12}} \\ \boldsymbol{k_2} & \boldsymbol{a_{22}} \end{vmatrix}}{\boldsymbol{D}} \quad \boldsymbol{and} \quad \boldsymbol{Y} = \frac{\begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{k_1} \\ \boldsymbol{a_{21}} & \boldsymbol{k_2} \end{vmatrix}}{\boldsymbol{D}}$$

The matrix of $\boldsymbol{D} = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} \end{vmatrix}$ is known as the coefficient matrix.

Example 3: Given **4*x* – 2*y* = 10** and **3*x* – 5*y* = 11**, solve for x and y using Cramer's rule.

Solution:

Step 1: Analyze.

Using the given definition of Cramer's rule, the equations of **4*x* – 2*y* = 10** and **3*x* – 5*y* = 11** yield the elements:

$$a_{11} = 4\,;\, a_{12} = -2\,;\, k_1 = 10$$
$$and$$
$$a_{21} = 3\,;\, a_{22} = -5\,;\, k_2 = 11$$

Step 2: Substitute and solve for the coefficient matrix.

$$\begin{aligned} D &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} \\ &= (4)(-5) - (3)(-2) \\ &= -20 - (-6) \\ &= -20 + 6 \\ &= -14 \end{aligned}$$

Step 3: Solve for X and Y.

$$\begin{aligned} X &= \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \\ &= \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} \\ &= \frac{(10)(-5) - (11)(-2)}{-14} \\ &= \frac{-50 + 22}{-14} \\ &= \frac{-28}{-14} = 2 \end{aligned}$$

$$\begin{aligned} Y &= \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D} \\ &= \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} \\ &= \frac{(4)(11) - (3)(10)}{-14} \\ &= \frac{44 - 30}{-14} \\ &= \frac{14}{-14} = -1 \end{aligned}$$

$$\therefore (x\,,\, y) = (2\,,\, -1)$$

Cramer's rule for solving a system of three equations and three unknowns is defined by:

$$\textit{Given}\ \begin{array}{l} a_{11}x + a_{12}y + a_{13}z = k_1 \\ a_{21}x + a_{22}y + a_{23}z = k_1 \\ a_{31}x + a_{32}y + a_{33}z = k_1 \end{array}\ \textit{with}\ D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

$$\textit{then}\ X = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{D},\ Y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{D},\ Z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{D}$$

The matrix D is known as the coefficient matrix.

Example 4: Use Cramer's rule to solve for the following system:

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x + 0y + \ \ z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

Solution:

Step 1: Analyze.

$$\begin{array}{llll} a_{11} = -1 & a_{12} = \ \ 2 & a_{13} = -3 & k_1 = 1 \\ a_{21} = \ \ 2 & a_{22} = \ \ 0 & a_{23} = \ \ 1 & k_2 = 0 \\ a_{31} = \ \ 3 & a_{32} = -4 & a_{33} = \ \ 4 & k_3 = 2 \end{array}$$

Step 2: Find the coefficient matrix, D.

$$\begin{aligned} D &= \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} \\ &= (-1)(-1)^2 \begin{vmatrix} 0 & 1 \\ -4 & 4 \end{vmatrix} + (2)(-1)^3 \begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} + (3)(-1)^4 \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} \\ &= (-1)(1)[(0)-(-4)] + (2)(-1)[(8)-(12)] + (3)(1)[(2)-(0)] \\ &= (-1)(4) + (-2)(-4) + (3)(2) \\ &= -4 + 8 + 6 \\ &= 10 \end{aligned}$$

Example 4 (Continued):

Step 3: Solve for x, y and z.

$$x=\frac{\begin{vmatrix}1 & 2 & -3\\ 0 & 0 & 1\\ 2 & -4 & 4\end{vmatrix}}{10}$$

$$=\frac{(-3)(-1)^4\begin{vmatrix}0 & 0\\ 2 & -4\end{vmatrix}+(1)(-1)^5\begin{vmatrix}1 & 2\\ 2 & -4\end{vmatrix}+(4)(-1)^6\begin{vmatrix}1 & 2\\ 0 & 0\end{vmatrix}}{10}$$

$$=\frac{(-3)(1)\left[(0)-(0)\right]+(1)(-1)\left[(-4)-(4)\right]+(4)(1)\left[(0)-(0)\right]}{10}$$

$$=\frac{(-3)(0)+(-1)(-8)+(4)(0)}{10}$$

$$=\frac{0+8+0}{10}$$

$$=\frac{8}{10}=\frac{4}{5}$$

$$y=\frac{\begin{vmatrix}-1 & 1 & -3\\ 2 & 0 & 1\\ 3 & 2 & 4\end{vmatrix}}{10}$$

$$=\frac{(-3)(-1)^4\begin{vmatrix}2 & 0\\ 3 & 2\end{vmatrix}+(1)(-1)^5\begin{vmatrix}-1 & 1\\ 3 & 2\end{vmatrix}+(4)(-1)^6\begin{vmatrix}-1 & 1\\ 2 & 0\end{vmatrix}}{10}$$

$$=\frac{(-3)(1)\left[(4)-(0)\right]+(1)(-1)\left[(-2)-(3)\right]+(4)(1)\left[(0)-(2)\right]}{10}$$

$$=\frac{(-3)(4)+(-1)(-5)+(4)(-2)}{10}$$

$$=\frac{-12+5-8}{10}$$

$$=-\frac{15}{10}=-\frac{3}{2}$$

Example 4 (Continued):

$$z=\frac{\begin{vmatrix}-1 & 2 & 1\\ 2 & 0 & 0\\ 3 & -4 & 2\end{vmatrix}}{10}$$

$$=\frac{(-1)(-1)^2\begin{vmatrix}0 & 0\\ -4 & 2\end{vmatrix}+(2)(-1)^3\begin{vmatrix}2 & 1\\ -4 & 2\end{vmatrix}+(3)(-1)^4\begin{vmatrix}2 & 1\\ 0 & 0\end{vmatrix}}{10}$$

$$=\frac{(-1)(1)\left[(0)-(0)\right]+(2)(-1)\left[(4)-(-4)\right]+(3)(1)\left[(0)-(0)\right]}{10}$$

$$=\frac{(-1)(0)+(-2)(8)+(3)(0)}{10}$$

$$=\frac{0-16+0}{10}$$

$$=-\frac{16}{10}=-\frac{8}{5}$$

Step 4: Analyze.

The solutions found were $x=\frac{4}{5}, y=-\frac{3}{2}\ and\ z=-\frac{8}{5}$.

This indicates that the solution set or point of interception for the three given lines is:

$$(x,y,z)=\left(\frac{4}{5},-\frac{3}{2},-\frac{8}{5}\right)$$