Matrices and Systems of Linear Equations

This section will explore the concept of the ***matrix*** and explain its use in expressing and solving systems of linear equations. A matrix (plural: *matrices*) is simply a rectangular array of symbols or numbers, all enclosed in large brackets. An example of a simple matrix would look like

$$\begin{bmatrix} 7 & 1 & 25 & 9 \\ 5 & 0 & 4 & 12 \\ 8 & 21 & 11 & 5 \end{bmatrix}$$

A matrix may have an equal number of rows and columns (in which case it is called a *square matrix*), but it is not required to. It may also have just a single row or column, such as

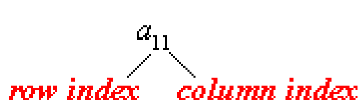
$$\begin{bmatrix} a & b & c & d & e \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$$

The only requirement is that the matrix be rectangular, with any number of rows and columns. Each number or symbol in the matrix is called an *element* of the matrix.

If the matrix is composed of a set of letters or symbols with two subscripts, such as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the first subscript is called the *row index*, denoting the row number of the matrix, and the second subscript is the *column index*, denoting the column number of the matrix.



Matrices can be used as a simplified way of writing a set of linear equations. In one method called an ***augmented matrix***, a vertical line is placed inside the matrix to represent a series of equal signs and dividing the matrix into two sides.

Thus, for example, the matrix

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & -3 \\ -1 & 1 & 2 & 1 \\ 5 & -2 & -3 & 7 \end{array}\right]$$

would represent the set of linear equations

$$\begin{array}{rcrcrcr} 2x & - & 3y & + & 4z & = & -3 \\ -x & + & y & + & 2z & = & 1 \\ 5x & - & 2y & - & 3z & = & 7 \end{array}$$

In the above augmented matrix, each row represents an equation. The numbers in the left side of the matrix represent the coefficients of the variables in the set of equations. The numbers in the right side of the matrix represent the constant values to the right of the equal signs. Note that if one or more of the variables did not exist in a particular equation, the coefficient associated with that variable would be zero, and a 0 would appear at that position in the matrix.

Instead of using augmented matrices, another way of writing a set of linear equations uses what are called ***matrix equations***. A matrix equation consists of three matrices: the first represents the values of the coefficients of the variables, the second lists the symbols of the variables themselves, and following an equal sign the third matrix represents the constant values to the right of the equal signs in the equations.

Thus, for the example of the set of linear equations above, the matrix equation for this set would look like

$$\begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 2 \\ 5 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix}$$

Example 1: Write the system of linear equations as an augmented matrix and as a matrix equation.

$$\begin{array}{rcrcrcr} & & x & + & z & = & 3 \\ 4x & + & 2y & + & z & = & 8 \\ & & 3x & + & y & = & 10 \end{array}$$

Example 1 (Continued):

Solution:

Add a zero coefficient for any missing variable terms

$$
\begin{aligned}
x &+ 0y + z = 3 \\
4x &+ 2y + z = 8 \\
3x &+ y + 0z = 10
\end{aligned}
$$

Augmented matrix:

$$
\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 4 & 2 & 1 & 8 \\ 3 & 1 & 0 & 10 \end{array}\right]
$$

Matrix equation:

$$
\begin{bmatrix} 1 & 0 & 1 \\ 4 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix}
$$