## Sequences and Series

A sequence is simply a list of numbers; each number is called a *term*. The first term is often called $a_1$, which is read "$a$ sub 1." In this case, the second term is called $a_2$. One way to represent the first $n$ terms of a sequence is this: $a_1, a_2, a_3, \ldots a_n$

## Arithmetic Sequences

A sequence in which consecutive terms differ by the same amount is called an *arithmetic sequence*. Illustration: 7, 10, 13, 16…304. Notice that the difference between consecutive terms is always 3. One way to describe this sequence is to say "The first number is 7. The rest of the numbers are formed by adding 3 to the previous number until you get to 304." This description is called a *recursive formula* for the sequence. In symbols, the recursive formula looks like this:

$a_1 = 7$ (which means that the first term is 7.)

$a_n = a_{n-1} + 3$, for $n \geq 2$ (which means: if $n \geq 2$, then the $n$th term is the $(n - 1)$ term + 3.)

If we call the difference between consecutive terms "$d$," then the recursive formula can be written thus:

| Recursive Formula for an arithmetic sequence |
|---|
| $a_1 = a$ (The first term is some number we'll call $a$.)<br><br>$a_n = a_{n-1} + d$, for $n \geq 2$ (If $n \geq 2$, then the $n$th term is the $(n - 1)$ term + $d$.) |

Notice that, if the arithmetic sequence, 7, 10, 13, 16…304 has $n$ terms, it can be written like this:

$7, (7 + 3), (7 + 2 \cdot 3), (7 + 3 \cdot 3), (7 + 4 \cdot 3) \ldots (7 + x \cdot 3)$

where $x$ is some unknown number. To find $x$, note that the *second* term is $(7 + 1 \cdot 3)$, the *third* term is $(7 + 2 \cdot 3)$, and so on, so that the $n$th term is $7 + (n - 1) \cdot 3$, which allows us to determine the value of the $n$th term for any value of $n$. This is reflected in what is called the *explicit formula* for the $n$th term of an arithmetic sequence.

| Explicit formula for the $n$th term of an arithmetic sequence |
|---|
| $a_n = a_1 + (n - 1)d$ |

Let's see if we can use the explicit formula to determine how many numbers are in the sequence:

7, 10, 13, 16…304

If there are $n$ terms in this sequence, then the last term is $a_n$

We can use the explicit formula to determine $n$ by solving for $n$ in the equation:

$a_n = a_1 + (n-1)d$ where $a_n = 304$, $a_1 = 7$ and $d = 3$.

Verify that if $304 = 7 + (n-1)3$, then $n = 100$.

In general, we can determine the number of terms in an arithmetic sequence thus:

To determine the number of terms in a finite arithmetic sequence, solve for $n$ in the explicit formula: $a_n = a_1 + (n-1)d$

Thus, if we know the values of $a_n$, $a_1$, and $d$, then the number of terms in the sequence is given by:

$$n = \frac{a_n - a_1}{d} + 1$$

## Arithmetic Series

If we add the terms in a sequence, we get a series. Let us develop a formula for adding the first $n$ terms of an arithmetic series.

Using the previous sequence

7, 10, 13, 16…304

we form the series: $S = 7 + 10 + 13 + 16 + \ldots + 304$.

Let's see if we can determine the sum, $S$, without adding the numbers one at a time.

Notice that if we write the terms in $S$ backwards, we get the same sum. That is,

$S = 7 + 10 + 13 + 16 + \ldots + 304$

$S = 304 + 300 + 297 + 294 + \ldots + 7$

Let's arrange the terms in both of these sums as follows:

| | | | | | |
|---|---|---|---|---|---|
| 7 | 10 | 13 | 16 | … | 304 |
| 304 | 301 | 298 | 295 | … | 7 |

Notice that if we add all of the numbers in both rows together, the sum will be 2*S*. (Why?) Instead of adding from left to right, let us add *vertically*.

| | | | | | |
|---|---|---|---|---|---|
| 7 | 10 | 13 | 16 | … | 304 |
| 304 | 301 | 298 | 295 | … | 7 |
| 311 | 311 | 311 | 311 | … | 311 |

If we add all of the 311's in the last row, the total must be 2*S*. The number of 311's is just *n*, which is 100, the number of terms in the series. (Why?) Thus, the sum of these *n* terms is given by $\frac{311(100)}{2} = 15550$.

In general, if we let $A_n$ represent the sum of the first *n* terms in an arithmetic series, then

| Sum of the first *n* terms in an arithmetic series | |
|---|---|
| $A_n = \frac{(a_1 + a_n)n}{2}$ | where $n = \frac{a_n - a_1}{d} + 1$ |

Geometric Sequences

A *geometric sequence* is a sequence in which the ratio of two consecutive terms is a constant.

| Geometric sequence, with $a_1 = a$ (the first term) | Illustration |
|---|---|
| $a, ar, ar^2, ar^3, \ldots ar^{n-1}$ | 3, 12, 48, 192…12288 (Here, the ratio is 4) |

| Recursive formula for a geometric sequence |
|---|
| $a_1 = a$ (the first term)<br><br>$a_n = a_{n-1} \cdot r$ for $n \geq 2$ |

| Explicit formula for the *n*th term of geometric sequence |
|---|
| $a_n = a_1 r^{n-1}$ |

# Geometric Series

If we add the terms of a geometric sequence, we get a geometric series.

(1) $S = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-1}$

To sum the series, let us first multiply both sides of (1) by $r$:

(2) $rS = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + ... + a_1 r^n$

Notice that there is only one term in (1) that is not in (2), and only one term in (2) that is not in (1). Thus, if we subtract (1) from (2), most of the terms will cancel out, leaving:

$$rS - S = a_1 r^n - a_1$$

which we now solve for $S$:

$$rS - S = a_1 r^n - a_1$$
$$S(r-1) = a_1(r^n - 1)$$
$$S = \frac{a_1(r^n - 1)}{r - 1}$$

If we let $A_n$ represent the sum of the first $n$ terms in a geometric series, we get:

| Sum of first $n$ terms in a geometric series |
|---|
| $A_n = \frac{a_1(r^n - 1)}{r - 1}$ <br> Where $a_1$ is the first term, $r$ is the ratio, and $n$ is the number of terms. |

Note that the last term in equation (2) above, $a_1 r^n$, was obtained by multiplying the last term of the series, $a_n$, by $r$. Hence, it is possible to determine the sum of a geometric series without knowing the number of terms, as in the following alternate formula.

| Alternate formula for the sum of first $n$ terms in a geometric series |
|---|
| $A_n = \frac{a_1 r^n - a_1}{r - 1} = \frac{a_n r - a_1}{r - 1}$ <br> Where $a_n$ is the last term, $a_1$ is the first term, and $r$ is the ratio. |

Illustration:

$S = 3 + 12 + 48 + 192 + \ldots + 12288$

By the alternate formula, we have $S = \dfrac{12288(4) - 3}{4 - 1} = 16383$

If we wish to determine $n$, we may solve for $n$ in the explicit formula thus:

$a_n = a_1 r^{n\text{-}1} = 3 \cdot 4^{n\text{-}1} = 12288$

$4^{n\text{-}1} = 4096$

$(n - 1)\log 4 = \log 4096$

$n - 1 = \dfrac{\log 4096}{\log 4} = \dfrac{3.612359948}{0.6020599913} = 6$ (believe it or not!)

$n = 7$

Now we can calculate the sum $S = 3 + 12 + 48 + 192 + \ldots + 12288$ by using the formula:

$$A_n = \frac{a_1(r^n - 1)}{r - 1} = \frac{3(4^7 - 1)}{4 - 1} = 16383$$

## Infinite Geometric Series

It turns out that, if the ratio, $r$, in an *infinite* geometric series is between -1 and 1, then such an infinite series has a finite sum. Recall the formula for the sum of a finite geometric series:

$$A_n = \frac{a_1(r^n - 1)}{r - 1}$$

This formula can be rewritten:

$$A_n = \frac{a_1(1 - r^n)}{1 - r}$$

which is convenient when $r$ lies between -1 and 1, in which case, as $n$ increases without limit, the term, $r^n$, becomes arbitrarily close to 0. Hence, we have:

| Sum of infinite geometric series, with $-1 < r < 1$ |
|---|
| $A = \dfrac{a_1}{1 - r}$ |

# Sigma Notation

The following is read: "The sum from $i = 1$ to 1000 of $10i - 4$.

$$\sum_{i=1}^{1000} 10i - 4$$ where "$10i - 4$" means "10 times the number $i$ minus 4."

What is required is to add all of the numbers generated by replacing $i$ with 1, then 2, then 3, etc. until we get to 1000.

To see some of the numbers we are to add:

- First replace $i$ with 1 obtaining $10 - 4 = 6$.
- Then replace $i$ with 2, obtaining $20 - 4 = 16$.
- Then replace $i$ with 3, obtaining $30 - 4 = 26$.
- Continue until the last value of $i$, which is 1000.

Hence

$$\sum_{i=1}^{1000} 10i - 4 = 6 + 16 + 26 + \ldots + 9996$$

We now see that this is an *arithmetic* series which can be added by using the formula on page D3.

Similarly, $\sum_{i=0}^{14} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^{14}$

Recall that

$2^0 = 1$

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

etc.

We now see that this is a *geometric* series which can be added by using the formula on page 31C.

Consider for example the infinite geometric series: $A = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots +$ where the ratio, $r$, and the first term, $a_1$, are both $\frac{1}{2}$

Here, $A = \dfrac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

Sigma Notation

The Greek letter capital sigma is sometimes used to define series. Example:

$$\sum_{k=1}^{100} (3k + 4)$$

This is read, "The sum from $k = 1$ to 100 of $3k + 4$." Here's what it means:

evaluate $3k + 4$ for $k = 1, 2, 3, \ldots, 100$ and add the results. That is,

$$\sum_{k=1}^{100} (3k + 4) = 7 + 10 + 13 + 16 + \ldots + 304$$

(You may recognize this series from our previous work.)

Problems.

| | Series | Is it arithmetic, geometric, or neither? | What is the sum? |
|---|---|---|---|
| 1 | 12 + 17 + 22 + 27 + . . . + 50,007 | | |
| 2 | 12 + 48 + 192 + 768 + . . . + 805306368 | | |
| 3 | 7 + 12 + 17 + 22 + 27 + . . . + 50,002 | | |
| 4 | A set of tires for an 18-wheeler will be replaced every year for the next 10 years. If the tires currently cost $10,000, and if they will increase in cost by 4% each year, what is the total amount that will be spent on these tires over the 10-year period? | | |
| 5 | Suppose a couple spends $500 per month on food. If during the next ten years the cost of food increases at an annual rate of 6%, what is the total amount this couple will need to spend on food over the next decade if their eating habits remain the same? | | |

D8

Name________________________________________

Find the sum of each of the following series.

| | |
|---|---|
| 1. $6 + 16 + 26 + 36 + \ldots + 9996$ | 6. $\sum_{k=0}^{14} \frac{1}{2^k}$ |
| 2. $\sum_{i=1}^{1000} 10i - 4$ | 7. $\sum_{j=1}^{10^{12}} j$ |
| 3. $\sum_{k=1}^{100} 5k + 3$ | 8. $\sum_{k=1}^{\infty} \frac{1}{2^k}$ |
| 4. $1 + 2 + 4 + 8 + 16 + \ldots + 16384$ | 9. $\sum_{k=1}^{100} \frac{3}{10^k}$ |
| 5. $\sum_{i=0}^{14} 2^i$ | 10. $\sum_{k=1}^{\infty} \frac{3}{10^k}$ |

11. ____________________ Suppose a couple spends $500 per month on food. If during the next ten years the cost of food increases at an annual rate of 6%, what is the total amount this couple will need to spend on food over the next decade if their eating habits remain the same?

D9

Name___________________________________

| Series | Is it arithmetic, geometric, or neither? | What is the sum? |
|---|---|---|
| 7 + 12 + 17 + 22 + 27 + . . . + 50,002 | | |
| 3 + 12 + 48 + 192 + 768 + . . . + 805306368 | | |

A set of tires for an 18-wheeler will be replaced every year for 10 years beginning in 2003. If the tires cost $10,000 in 2003 and if they will increase in cost by 5% each year, what is the total amount that will be spent on these tires over the 10-year period?

2003: $10,000

2004 __________

2005 __________

.

.

.

2012 __________

Assume that the earth is in a circular orbit 93,000,000 miles from the sun and that a year is 365.25 days. What is the speed of the earth relative to the sun? Use 3.14 for $\pi$. Give the answer in:

__________miles per year

__________miles per day

__________miles per hour

__________miles per minute

__________miles per second

Name________________________________

| Series | Is it arithmetic, geometric, or neither? | What is the sum? |
|---|---|---|
| 7 + 12 + 17 + 22 + 27 + . . . + 50,002 | | |
| 3 + 12 + 48 + 192 + 768 + . . . + 805306368 | | |

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.

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2012 __________

Assume that the earth is in a circular orbit 93,000,000 miles from the sun and that a year is 365.25 days. What is the speed of the earth relative to the sun? Use 3.14 for π. Give the answer in:

___________miles per year

___________miles per day

___________miles per hour

___________miles per minute

___________miles per second

Suppose you borrow \$100,000 at 6% annual interest rate, and your monthly payments are \$600. The interest you owe for the first month is one-twelfth of 6% of 100,000, i.e., $0.06(100{,}000) \div 12 = \$500$. (Discuss: Why one-twelfth?) Notice that we get the same answer if we first divide 0.06 by 12 and then multiply by 100,000: $0.06 \div 12 = 0.005$ and $0.005(100{,}000) = 500$. The 0.005 can be thought of as the "monthly interest rate."

This summarizes the situation after the first payment:

| Payment number and amount | Interest Payment | Principal Payment | Balance |
|---|---|---|---|
| 1. \$600 | \$500 | \$100 | \$99,900 |

Now that the new balance is \$99,900, here are the relevant numbers after the second payment. Can you explain the rationale for each calculation?

Interest payment: $0.005(\$99{,}900) = \$499.50$
Principal payment: $\$600 - \$499.50 = \$100.50$
New balance: $\$99{,}900 - 100.50 = \$99{,}799.50$

In tabular form, it looks like this:

| Payment number and amount | Interest Payment | Principal Payment | Balance |
|---|---|---|---|
| 1. \$600 | \$500 | \$100 | \$99,900 |
| 2. \$600 | \$499.50 | \$100.50 | \$99,799.50 |

Typically, this process continues until the balance is zero. Notice that, over the life of the loan, the interest payment goes down while the principal payment goes up. Eventually, the interest payment is close to zero while the principal payment approaches \$600. Such a list of the interest payments, principal payments, and balances is often called an "amortization schedule" or "amortization table." (To "amortize" the loan means to kill it! That comports with the expression, "over the *life* of the loan.)

Let's see if we can derive a formula that one could use to calculate the payment necessary to repay a loan of $P$ dollars in $n$ equal monthly payments at monthly interest rate $i$, where $i$ is one-twelfth of the annual interest rate.

Call the monthly payment $M$. The first payment must include the interest due, which is $P$ times $i$, or $Pi$. The rest of the payment, $M - Pi$, reduces the balance to:

$$P - (M - Pi) = P - M + Pi = P + Pi - M = P(1 + i) - M$$

Hence, after the first payment, we have:

| Payment | Interest Payment | Principal Payment | Balance |
|---|---|---|---|
| 1. $M$ | $Pi$ | $M - Pi$ | $P(1 + i) - M$ |

Notice that the expression for the new balance, $P(1 + i) - M$, could have been derived in another way. When the first payment is due, the amount owed is the original principal, $P$, plus the interest on that principal, $Pi$. That is, the amount owed is:
$P + Pi = P(1 + i)$. After a payment of $M$ dollars has been made, the amount owed is reduced by $M$ dollars, so that the new balance must be $P(1 + i) - M$. Neat!

In sum, after each payment has been made, the new balance will be the monthly payment, $M$, subtracted from the product of the old balance and $(1 + i)$. After the first payment, the new value of the "old balance" would be $P(1 + i) - M$. Thus, after the second payment, we would have:

new balance = $[P(1 + i) - M](1 + i) - M$ (Why?)

new balance = $P(1 + i)^2 - M(1 + i) - M$ (How was this done?)

Similarly, it can be shown that, after the third payment, the new balance will be:

$$P(1 + i)^3 - M(1 + i)^2 - M(1 + i) - M$$

Notice that the last three terms of this expression form a geometric series, viz.,

$$-M(1 + i)^2 - M(1 + i) - M$$

For convenience, let us write the series in reverse order:

$$-M - M(1 + i) - M(1 + i)^2$$

$$= -M[1 + (1 + i) + (1 + i)^2]$$

In this form, it may be easier to recognize the geometric series in the brackets. Recall the formula for the sum of the first $n$ terms of a geometric series:

$A_n = \dfrac{a_1(r^n - 1)}{r - 1}$ where $a_1$ is the first term, $r$ is the "multiplier" that characterizes a geometric series, and $n$ is the number of terms.

Substituting into this formula, we find that

$$-M[1 + (1 + i) + (1 + i)^2] = -M\frac{1 \cdot [(1+i)^3 - 1]}{1 + i - 1} = -M\frac{(1+i)^3 - 1}{i}$$

Thus, the balance after three payments is: $P(1+i)^3 - M\dfrac{(1+i)^3 - 1}{i}$

Before looking at the next page, can you predict the balance after $n$ payments?

The balance after $n$ payments is given by the formula:

$$P(1+i)^n - M\frac{(1+i)^n - 1}{i}$$

Now, recall that we are trying to determine the value of $M$ that will reduce the balance to zero in $n$ payments. Hence, we now set the above expression equal to zero and solve for $M$. Provide an explanation for each of these steps.

| Equation | Explanation |
|---|---|
| $P(1+i)^n - M\frac{(1+i)^n - 1}{i} = 0$ | This is the equation we are to solve for $M$. |
| $P(1+i)^n = M\frac{(1+i)^n - 1}{i}$ | |
| $Pi(1+i)^n = M[(1+i)^n - 1]$ | |
| $M = \frac{Pi(1+i)^n}{(1+i)^n - 1}$ | |
| Note that the expression $(1+i)^n$ occurs twice in this formula. Let's divide the numerator and denominator by $(1+i)^n$. Again, provide explanations for these last two steps. | |
| $M = \frac{Pi}{1 - \frac{1}{(1+i)^n}}$ | |
| $M = \frac{Pi}{1-(1+i)^{-n}}$ | This is the way the formula is generally written. |

Note. Because $i$ was used to represent the *monthly* interest rate, it was calculated to be the *annual* interest rate divided by 12. While *monthly* payments are a way of life for many of us, the formula works regardless of the interval between payments: the ***periodic*** interest rate, $i$, is the *annual interest rate* divided by the *number of payments made per year*. (The intervals between consecutive payments are often not quite uniform, but this fact is generally ignored.)

Formulas like the one we just derived are available on the Internet. The average, non-mathematically inclined person can simply enter a few numbers and get valuable information that used to be accessible only to those with specialized knowledge. Thus, most of us can easily and painlessly utilize in our everyday lives a really powerful and important mathematical result.

Before you complete the Internet assignment below, answer these two questions.

1. Do you think that the monthly payment required to repay a $100,000 note in 30 years (360 months) would be twice as much as the payment required to repay it in 15 years (180 months), or would it be more or less than twice as much? (Although the annual interest rate is irrelevant, assume it's 6% in both cases.)

2. Do you think that repaying a single loan of $100,000 would require a monthly payment greater than, less than, or the same as the *total* of two monthly payments required to repay separate notes of $75,000 and $25,000? (Again, the annual interest rate is irrelevant, but assume it's 8% for this question.)

Now use the following site to fill in the blanks in the three rightmost columns of the table below.

http://ray.met.fsu.edu/~bret/amortize.html

(CAUTION: Before each new calculation, erase the number that the computer has calculated and displayed in the box labeled "Payment Amount.")

| Principal | Interest rate | Number of payments per year | Total number of payments | Payment amount | Total of all payments | Total amount of interest paid |
|---|---|---|---|---|---|---|
| | | | | | | |
| $100,000 | 6% | 12 | 360 | | | |
| | | | | | | |
| $100,000 | 6% | 12 | 180 | | | |
| | | | | | | |
| $75,000 | 8% | 12 | 72 | | | |
| $25,000 | 8% | 12 | 72 | | | |
| | | | | | | |
| $100,000 | 8% | 12 | 72 | | | |

To get a second opinion, (a) go to http://www.rbfcu.org/ (b) click first on "Loan Products," then on "Loan Calculator." Do you get the same results?

Compare your answers to the two questions above with the information you recorded in the table. Did you learn anything useful? ☺

Search the Internet using the word "Amortization." How many "hits" do you get? Can you find the site you used to complete the table above?

Hopefully, this exercise illustrates the fact that, even if we don't fully understand mathematics, we can appreciate its value and impact in our everyday lives.

D15

Recall the Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13…

A recursive formula is:

$a_1 = 1$
$a_2 = 1$
$a_n = a_{n-1} + a_{n-2}$ for $n > 2$.

Show that an explicit formula for the *nth* term of the Fibonacci Sequence is given by:

$$t_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

See if you can uncover a relationship between the two rightmost columns. Hint: each number in the right column is "close" to a number in the middle column, but on a different row.

| $n$ | $t_n$ | $\sum_{k=1}^{n} t_k$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 2 |
| 3 | 2 | 4 |
| 4 | 3 | 7 |
| 5 | 5 | 12 |
| 6 | 8 | 20 |
| 7 | 13 | 33 |
| 8 | 21 | 54 |
| 9 | 34 | 88 |
| 10 | 55 | 143 |
| 11 | 89 | 232 |
| 12 | 144 | 376 |
| 13 | 233 | 609 |
| 14 | 377 | 986 |
| 15 | 610 | 1596 |
| 16 | 987 | 2583 |
| 17 | 1597 | 4180 |

Show that the sum of the first $n$ terms of the Fibonacci Sequence is one less than the $n + 2^{nd}$ term. In symbols,

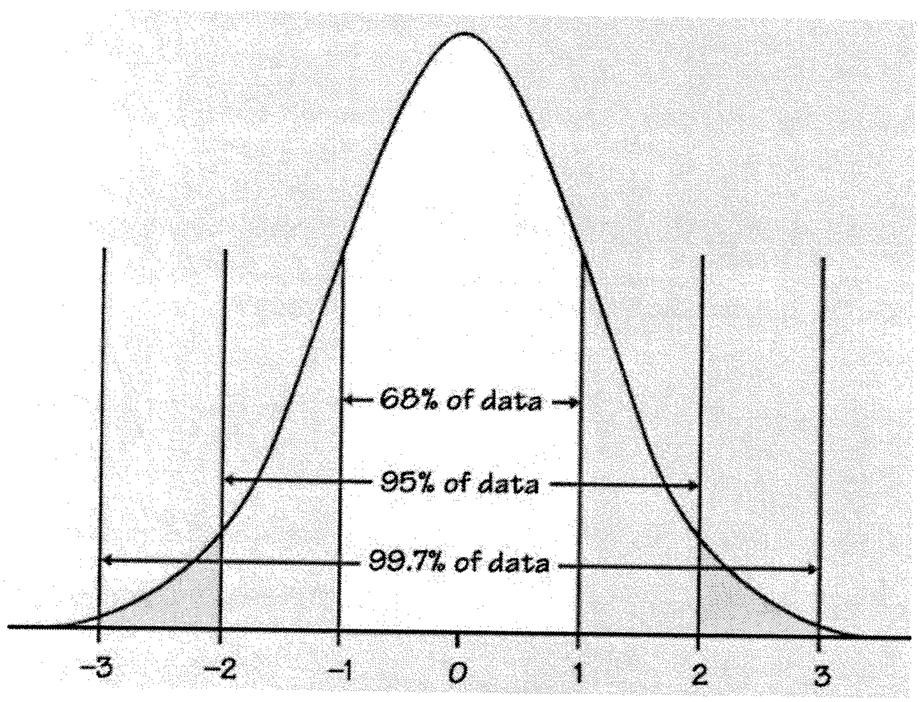
$$\sum_{k=1}^{n} t_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right] - 1$$

32D

# The z-score

## The Standard Normal Distribution

| Definition of the Standard Normal Distribution |
| --- |
| The *Standard Normal* distribution follows a normal distribution and has mean 0 and standard deviation 1 |




Notice that the distribution is perfectly symmetric about 0.

If a distribution is normal but not standard, we can convert a value to the Standard normal distribution table by first by finding how many standard deviations away the number is from the mean.

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## The z-score

The number of standard deviations from the mean is called the *z-score* and can be found by the formula

E 2

$$z = \frac{x - \mu}{\sigma}$$

**Example**

Find the z-score corresponding to a raw score of 132 from a normal distribution with mean 100 and standard deviation 15.

**Solution**

We compute

$$z = \frac{132 - 100}{15} = 2.133$$

---

**Example**

A z-score of 1.7 was found from an observation coming from a normal distribution with mean 14 and standard deviation 3. Find the raw score.

**Solution**

We have

$$1.7 = \frac{x - 14}{3}$$

To solve this we just multiply both sides by the denominator 3,

$$(1.7)(3) = x - 14$$

$$5.1 = x - 14$$

$$x = 19.1$$

---

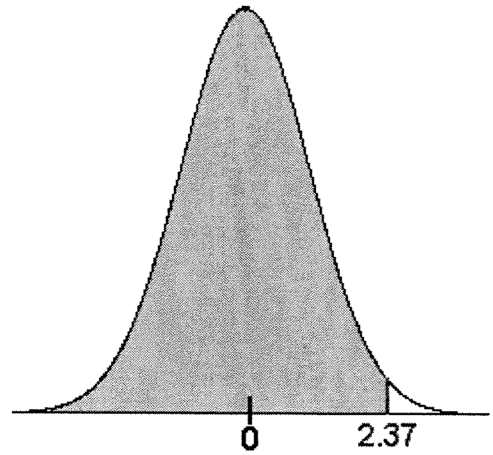
## The z-score and Area

Often we want to find the probability that a z-score will be less than a given value, greater than a given value, or in between two values. To accomplish this, we use the table from the textbook and a few properties about the normal distribution.

**Example**

Find

$P(z < 2.37)$




**Solution**

We use the table. Notice the picture on the table has shaded region corresponding to the area to the left (below) a z-score. This is exactly what we want. Below are a few lines of the table.

| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **2.2** | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| **2.3** | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | **.9911** | .9913 | .9916 |
| **2.4** | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |

The columns corresponds to the ones and tenths digits of the z-score and the rows correspond to the hundredths digits. For our problem we want the row 2.3 (from 2.37) and the row .07 (from 2.37). The number in the table that matches this is .9911.

Hence

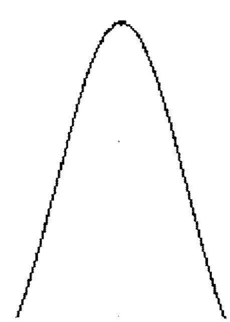
$P(z < 2.37) = .9911$

---

**Example**

Find

$P(z > 1.82)$



**Solution**

In this case, we want the area to the right of 1.82. This is not what is given in the table. We can use the identity

$$P(z > 1.82) = 1 - P(z < 1.82)$$

reading the table gives

$$P(z < 1.82) = .9656$$

Our answer is

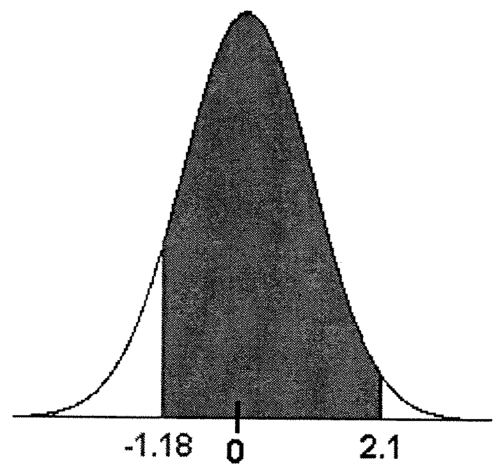
$$P(z > 1.82) = 1 - .9656 = .0344$$

---

**Example**

Find

$$P(-1.18 < z < 2.1)$$




**Solution**

Once again, the table does not exactly handle this type of area. However, the area between -1.18 and 2.1 is equal to the area to the left of 2.1 minus the area to the left of -1.18. That is

$$P(-1.18 < z < 2.1) = P(z < 2.1) - P(z < -1.18)$$

To find $P(z < 2.1)$ we rewrite it as $P(z < 2.10)$ and use the table to get

$$P(z < 2.10) = .9821.$$

The table also tells us that

$$P(z < -1.18) = .1190$$

Now subtract to get

$$P(-1.18 < z < 2.1) = .9821 - .1190 = .8631$$

---

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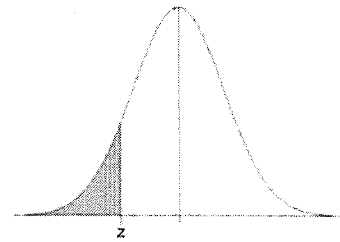
Back to the Elementary Statistics (Math 201) Home Page

# Standard Normal Cumulative Probability Table

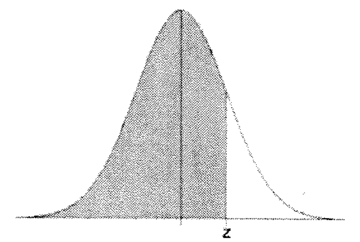



**Cumulative probabilities for NEGATIVE z-values are shown in the following table:**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| -3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| -3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| -3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| -3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| -3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| -2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| -2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| -2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| -2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| -2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| -2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| -2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| -2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| -2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| -2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| -1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| -1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| -1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| -1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| -1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| -1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| -1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| -1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| -1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| -1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| -0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| -0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| -0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| -0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| -0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| -0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| -0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| -0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| -0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| 0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |

E-6

# Standard Normal Cumulative Probability Table




**Cumulative probabilities for POSITIVE z-values are shown in the following table:**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| **3.1** | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| **3.2** | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| **3.3** | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| **3.4** | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

EC.1

Name_______________________________

First, calculate the sample mean, median, mode, range, and standard deviation for the following set of grades. Then calculate the $z$-score for each grade, rounded to two decimal places.

| $z$-score: $\frac{x_i - \bar{x}}{s}$ | $i$ | $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $n =$ ________<br>Mean: $\bar{x} = \frac{\sum x_i}{n} =$ |
|---|---|---|---|---|---|
| | 1 | 90 | | | Median = middle value = |
| | 2 | 85 | | | Mode = most frequent value = |
| | 3 | 85 | | | Range = high – low = |
| | 4 | 85 | | | Standard deviation:<br>$s = \sqrt{\sum \frac{(x_i - \bar{x})^2}{n-1}} =$ |
| | 5 | 75 | | | |
| | 6 | 75 | | | |
| | 7 | 70 | | | |
| | 8 | 65 | | | |
| | 9 | 54 | | | |
| | | $\sum x_i =$ | | $\sum (x_i - \bar{x})^2 =$ | |

EG.2

Name Key

First, calculate the sample mean, median, mode, range, and standard deviation for the following set of grades. Then calculate the *z*-score for each grade. Round to two decimal places.

| *z*-score: $\frac{x_i - \bar{x}}{s}$ | $i$ | $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $n =$ 9<br>Mean: $\bar{x} = \frac{\sum x_i}{n} =$ 76 |
|---|---|---|---|---|---|
| 1.20 | 1 | 90 | 14 | 196 | Median = middle value = 75 |
| 0.77 | 2 | 85 | 9 | 81 | Mode = most frequent value = 85 |
| 0.77 | 3 | 85 | 9 | 81 | Range = high – low = 36 |
| 0.77 | 4 | 85 | 9 | 81 | Standard deviation: |
| -.0859 | 5 | 75 | -1 | 1 | $s = \sqrt{\sum \frac{(x_i - \bar{x})^2}{n-1}} =$ |
| -.08598 | 6 | 75 | -1 | 1 | |
| -.5159 | 7 | 70 | -6 | 36 | $\sqrt{\frac{1082}{8}}$ |
| -0.9458 | 8 | 65 | -11 | 121 | $\sqrt{135.25}$ |
| -1.89 | 9 | 54 | -22 | 484 | 11.6297033 |
| | | $\sum x_i =$ 684 | | $\sum (x_i - \bar{x})^2 =$ 1082 | ≈ 11.63 |

E63

Name______________________________

First, calculate the sample mean, median, mode, range, and standard deviation for the following set of grades. Then calculate the $z$-score for each grade, rounded to two decimal places.

| $z$-score: $\frac{x_i - \bar{x}}{s}$ | $i$ | $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $n =$ ________ <br> Mean: $\bar{x} = \frac{\sum x_i}{n} =$ |
|---|---|---|---|---|---|
| | 1 | 90 | | | Median = middle value = |
| | 2 | 85 | | | Mode = most frequent value = |
| | 3 | 85 | | | Range = high – low = |
| | 4 | 85 | | | Standard deviation: |
| | 5 | 75 | | | |
| | 6 | 75 | | | $s = \sqrt{\sum \frac{(x_i - \bar{x})^2}{n-1}} =$ |
| | 7 | 70 | | | |
| | 8 | 67 | | | |
| | | $\sum x_i =$ | | $\sum (x_i - \bar{x})^2 =$ | |

E6.4

Name Key

First, calculate the sample mean, median, mode, range, and standard deviation for the following set of grades. Then calculate the $z$-score for each grade. Round to two decimal places.

| $z$-score: $\frac{x_i - \bar{x}}{s}$ | $i$ | $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ |
|---|---|---|---|---|
| 1.3205 ✓ | 1 | 90 | 11 | 121 |
| 0.7203 ✓ | 2 | 85 | 6 | 36 |
| .7203 | 3 | 85 | 6 | 36 |
| .7203 | 4 | 85 | 6 | 36 |
| -.4802 ✓ | 5 | 75 | -4 | 16 |
| -.4802 | 6 | 75 | -4 | 16 |
| -1.0804 ✓ | 7 | 70 | -9 | 81 |
| -1.4406 ✓ | 8 | 67 | -12 | 144 |
| | | $\Sigma x_i =$ 632 | | $\sum (x_i - \bar{x})^2 =$ 486 |

$n =$ 8 ✓

Mean: $\bar{x} = \frac{\Sigma x_i}{n} =$ 79 ✓

Median = middle value = 80 ✓

Mode = most frequent value = 85 ✓

Range = high – low = 23 ✓

Standard deviation:

$$s = \sqrt{\sum \frac{(x_i - \bar{x})^2}{n - 1}} =$$

$\sqrt{\frac{486}{7}}$

$\sqrt{69.42857}$

8.33238 ✓

≈ 8.33

E6.5

Name________________________________

While you might find this e-mail message humorous, how would you respond to someone who took it seriously?

More Statistics

a. The number of physicians in the US is 700,000.
b. Accidental deaths caused by Physicians per year is 120,000.
c. Accidental deaths per physician is 0.171(US Dept. of Health & Human Services)

Then think about this:

a. The number of gun owners in the US is 80,000,000 (yes, eighty-million!).
b. The number of accidental gun deaths per year (all age groups) is 1,500.
c. The number of accidental deaths per gun owner is .0000188.

Statistically, doctors are approximately 9,000 times more dangerous than gun owners.

FACT: NOT EVERYONE HAS A GUN, BUT ALMOST EVERYONE HAS AT LEAST ONE DOCTOR.

Please alert your friends to this alarming threat. We must ban doctors before this gets out of hand. As a public service, I have omitted the statistics on lawyers for fear that the shock could cause people to seek medical attention.

E6.6

# Sampling with M&M's

1. Without looking, take exactly 10 M&M's out of the bag and record the number of each color below:

| Color | Number |
|---|---|
| Red | |
| Orange | |
| Yellow | |
| Green | |
| Blue | |
| Brown | |

2. Based only on the ten M&M's you selected:

- Estimate the percentage of M&M's in the bag that are blue.
- Circle the color you would guess represents the smallest group in the bag.

  Red Orange Yellow Green Blue Brown

- Circle the color you would guess represents the second smallest group in the bag.

  Red Orange Yellow Green Blue Brown

- Circle the color you would guess represents the largest group in the bag.

  Red Orange Yellow Green Blue Brown

After you complete the reverse side of this paper, write down whatever occurs to you about this entire experience.

E67

Record the data gathered from each student.

| | Red | Orange | Yellow | Blue | Brown |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |
| 7 | | | | | |
| 8 | | | | | |
| 9 | | | | | |
| 10 | | | | | |
| 11 | | | | | |
| 12 | | | | | |
| 13 | | | | | |
| 14 | | | | | |
| 15 | | | | | |
| 16 | | | | | |
| 17 | | | | | |
| 18 | | | | | |
| 19 | | | | | |
| 20 | | | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | | | | | |
| 24 | | | | | |
| 25 | | | | | |
| 26 | | | | | |
| 27 | | | | | |
| 28 | | | | | |
| 29 | | | | | |
| 30 | | | | | |

3. Based on the above data:

- Estimate the percentage of M&M's in the bag that are blue.

- Circle the color you would guess represents the smallest group in the bag.

  Red Orange Yellow Green Blue Brown

- Circle the color you would guess represents the second smallest group in the bag.

  Red Orange Yellow Green Blue Brown

- Circle the color you would guess represents the largest group in the bag.

  Red Orange Yellow Green Blue Brown

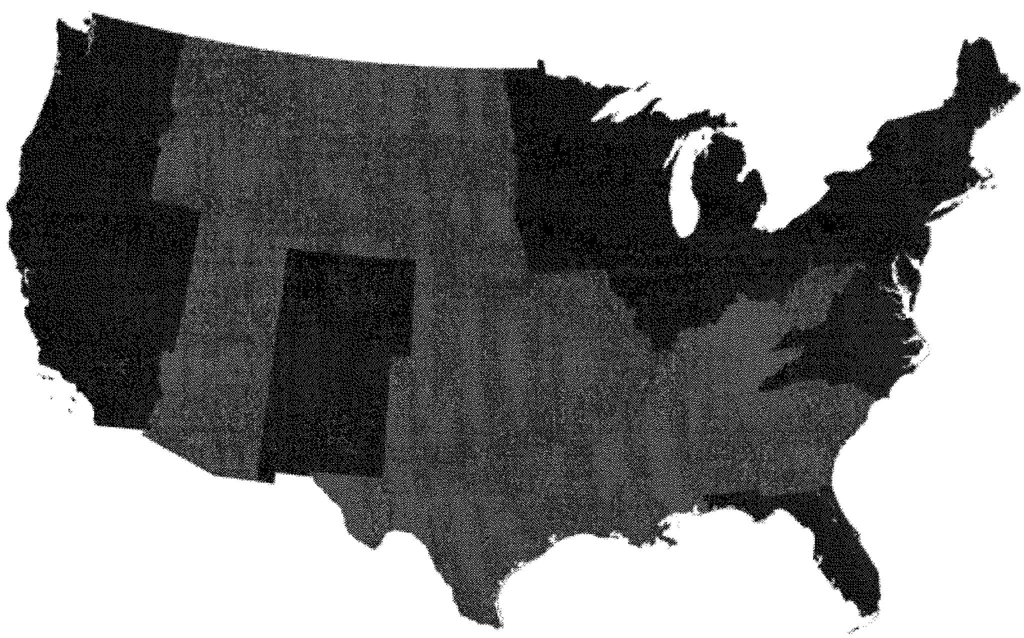
E7

# Maps of the 2008 US presidential election results

(Maps for the 2004 election are available here)

### Election results by state

Most of us are, by now, familiar with the maps the TV channels and web sites use to show the results of the presidential election:



The states are colored red or blue to indicate whether a majority of their voters voted for the Republican candidate, John McCain, or the Democratic candidate, Barack Obama, respectively.

## Judging by this map alone, who appears to have won the election, McCain (red) or Obama (blue)?

E 8

Looking at this map it gives the impression that the Republicans won the election handily, since there is rather more red on the map than there is blue. In fact, however, the reverse is true – the Democrats won by a substantial margin. The explanation for this apparent paradox, as pointed out by many people, is that the map fails to take account of the population distribution. It fails to allow for the fact that the population of the red states is on average significantly lower than that of the blue ones. The blue may be small in area, but they represent a large number of voters, which is what matters in an election.

We can correct for this by making use of a *cartogram*, a map in which the sizes of states are rescaled according to their population. That is, states are drawn with size proportional not to their acreage but to the number of their inhabitants, states with more people appearing larger than states with fewer, regardless of their actual area on the ground. On such a map, for example, the state of Rhode Island, with its 1.1 million inhabitants, would appear about twice the size of Wyoming, which has half a million, even though Wyoming has 60 times the acreage of Rhode Island.

Here are the 2008 presidential election results on a population cartogram of this type:

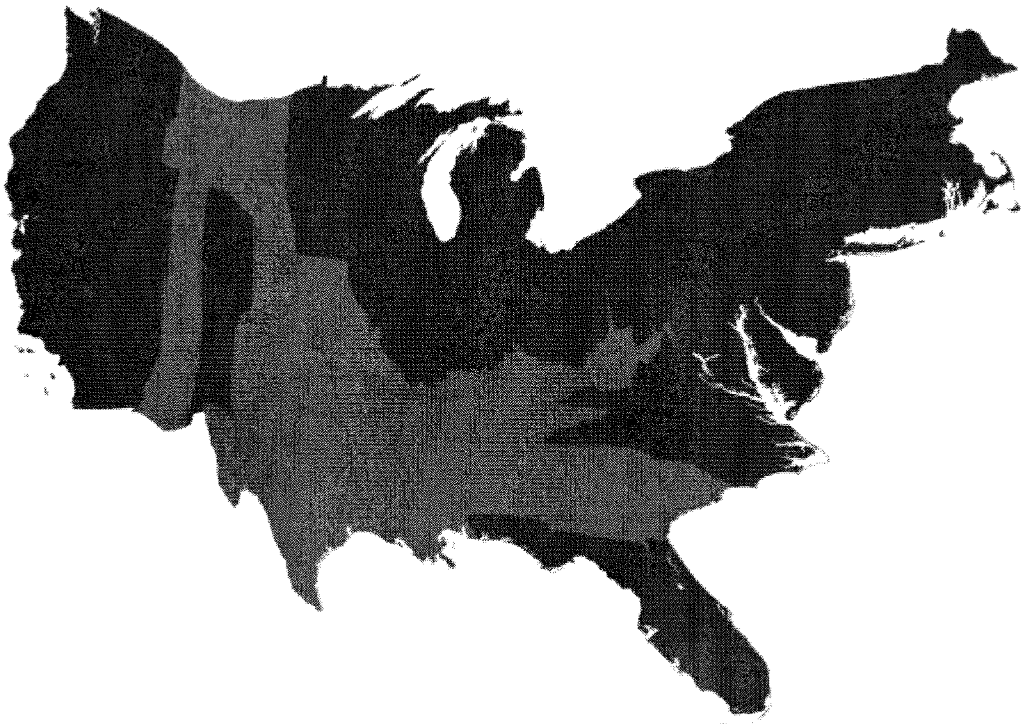
As you can see, the states have been stretched and squashed, some of them substantially, to give them the appropriate sizes, though it's done in such a way as to preserve the general appearance of the map, so far as that's possible. On this map there is now clearly more blue than red.

The presidential election, however, is not actually decided on the basis of the number of people who vote for each candidate but on the basis of the electoral college. Under the US electoral system, each state in the union contributes a certain number of electors to the electoral college, who vote according to the majority in their state. The candidate receiving a majority of the votes

E9

in the electoral college wins the election. The electors are apportioned roughly according to states' populations, as measured by the census, but with a small but deliberate bias in favor of smaller states.

We can represent the effects of the electoral college by scaling the sizes of states to be proportional to their number of electoral votes, which gives a map that looks like this:



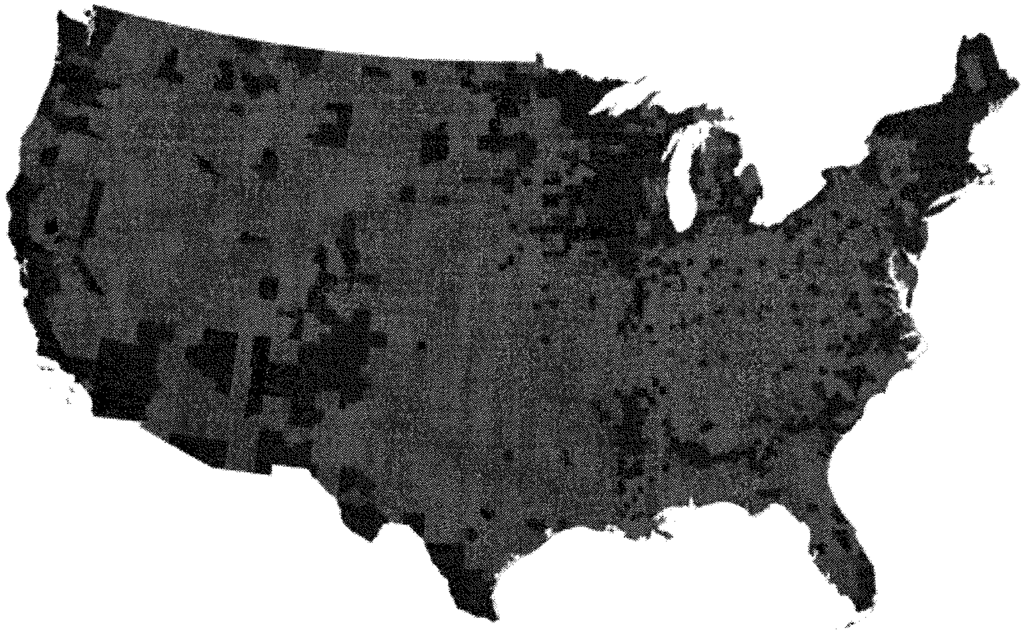
This cartogram looks similar to the one above it, but it's not identical. Wyoming, for instance, has approximately doubled in size, precisely because of the bias in favor of small states.

The areas of red and blue on the cartogram are now proportional to the actual numbers of electoral votes won by each candidate. Thus this map shows at a glance both which states went to which candidate and which candidate won more electoral college votes – something that you cannot tell easily from the normal election-night red and blue map.

## Election results by county

But we can go further. We can do the same thing also with the county-level election results and the images are even more striking. Here is a map of US counties, again colored red and blue to indicate Republican and Democratic majorities respectively:

E10



Now the effects we saw at the state level are even more pronounced: the red areas appear overwhelmingly in the majority, an appearance again at odds with the actual results of the election. Again, we can make a more helpful respresentation by using a cartogram. Here is what the cartogram looks like for the county-level election returns:

However, this map is still somewhat misleading because we have colored every county either red or blue, as if every voter voted the same way. This is of course not realistic: all counties contain both Republican and Democratic supporters and in using just the two colors on our map we lose

E 11

any information about the balance between them. There is no way to tell whether a particular county went strongly for one candidate or the other or whether it was relatively evenly split.

One way to improve the map and reveal more nuance in the vote is to use not just two colors, red and blue, but to use red, blue, and shades of purple in between to indicate percentages of votes. Here is what the normal map looks like if you do this:

And here's what the cartogram looks like:

E12

As this map makes clear, large portions of the country are quite evenly divided, appearing in various shades of purple, although a number of strongly Democratic (blue) areas are visible too, mostly in the larger cities. There are also some strongly Republican areas, but most of them have relatively small populations and hence appear quite small on this map.

A slight variation on the same idea is to use a nonlinear color scale like this:

These maps use a color scale that ranges from red for 70% Republican or more, to blue for 70% Democrat or more. This is sort of practical, since there aren't many counties outside that range anyway, but to some extent it also obscures the true balance of red and blue.

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## Notes:

**Frequently asked questions (FAQs):** A list of frequently asked questions concerning these maps, along with answers, can be found here.

E 13

**County results:** The county-level data I used came from here. For those who want to do their own analysis, here are spreadsheets of the complete county results in OpenDocument and Microsoft format. They are up-to-date as of November 16, 2008, but a small number of precincts still had not reported by that date, so a few results are missing.

**Correction:** Vote tallies for Richmond and Kings counties in New York were mistakenly interchanged on the original maps. This has now been corrected on all maps. Many thanks to C. Moore for pointing out the error.

**Shameless plug:** I have a new book of world cartograms entitled *The Atlas of the Real World*, coauthored with Daniel Dorling and Anna Barford. It's nothing to do with elections, but it contains more than 300 cartograms showing everything from who emits the most greenhouse gases to who imports the most fish. You can see a preview of some of the maps here. The book is supposed to be fun but also informative – the maps are a great way to learn many things about the world that you probably never knew. If you're interested in this stuff check it out.

**Poster:** A wall poster of these maps, made by the University of Minnesota, can be downloaded here.

**Software:** My computer software for producing cartograms is freely available here.

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